7. NEODREDJENI INTEGRALI

7.1 Opcenito o integralu i pravilima integriranja

Integriranje je inverzna racunska operacija od deriviranja.
Integrirati funkciju $f(x)$ znaci odrediti primitvnu funkciju $F(x)$ funkcije $f(x)$.

$\int f(x)dx = F(x)+C$ jer je $D_x\left[F(x)+C\right]=f(x)$ C je konstanta integracije.

Pravila integriranja:

1. $\int 0\ dx = C$
2. $\int 1\ dx = x+C$
3. $\int a\ dx = ax+C$
4. $\int x^n dx = \dfrac{x^{n+1}}{n+1}+C$; za sve racionalne brojeve; $n \neq -1$.
5. $\int a\,f(x)dx = a\int f(x)dx$
6. $\int\left[f(x)+g(x)\right]dx = \int f(x)dx+\int g(x)dx$
7. $\int\left[f(x)-g(x)\right]dx = \int f(x)dx-\int g(x)dx$
8. $\int\left[f(x)\right]^n f'(x)dx = \dfrac{1}{n+1}\left[f(x)\right]^{n+1}+C$ za sve racionalne brojeve; $n \neq -1$
9. $\int f(g(x))\,g'(x)dx = \int f(u)du$ metoda supstitucije
10. $\int u\ dv = uv-\int v\ du \Leftrightarrow \int f(x)g'(x)dx = f(x)g(x)-\int f'(x)g(x)dx$; metoda parcijalne integracije
11. Metoda parcijalni razlomaka

Primjeri primjene pravila integriranja:

1. $\int 0\ dx = C \quad \Rightarrow \int 0\ dx = C \Leftrightarrow D_x(C)=0$
2. $\int 1\ dx = x+C \quad \Rightarrow \int 1dx = x+C \Leftrightarrow D_x(x+C)=1$
3. $\int a\ dx = ax+C \quad \Rightarrow \int 7dx = 7x+C \Leftrightarrow D_x(7x+C)=7$
4. $\int x^n dx = \dfrac{x^{n+1}}{n+1}+C \quad \Rightarrow \int x^7 dx = \dfrac{x^8}{8}+C \Leftrightarrow D_x\left(\dfrac{x^8}{8}+C\right)=x^7$
5. $\int a\,f(x)dx = a\int f(x)dx \Rightarrow \int 5\sqrt[3]{x}dx = 5\int x^{\frac{1}{3}}dx = 5\dfrac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}+C = \dfrac{15}{4}x^{\frac{4}{3}}+C$

$$D_x\left(\frac{15}{4}x^{\frac{4}{3}}+C\right)=5x^{\frac{1}{3}}$$

6. $\int\left[f(x)+g(x)\right]dx=\int f(x)\,dx+\int g(x)\,dx$

$$\int\left(x^2+7\right)dx=\int x^2dx+\int 7dx=\frac{x^3}{3}+7x+C\Leftrightarrow D_x\left(\frac{x^3}{3}+7x+C\right)=x^2+7$$

7. $\int\left[f(x)-g(x)\right]dx=\int f(x)\,dx-\int g(x)\,dx$

$$\int\left(3x^3-5x\right)dx=\int 3x^3dx-\int 5xdx=\frac{3x^4}{4}-\frac{5x^2}{2}+C$$

$$D_x\left(\frac{3x^4}{4}-\frac{5x^2}{2}+C\right)=3x^3-5x$$

8. $\int\left[f(x)\right]^n f'(x)\,dx=\frac{1}{n+1}\left[f(x)\right]^{n+1}+C$ za sve racionalne brojeve; $n\neq-1$

$$\int\left(\frac{1}{3}x^3+7\right)^5x^2dx=\frac{1}{6}\left(\frac{1}{3}x^3+7\right)^6+C$$

$$D_x\left(\frac{1}{6}\left(\frac{1}{3}x^3+7\right)^6\right)=\left(\frac{1}{3}x^3+7\right)^5x^2$$

9. $\int f\left(g(x)\right)g'(x)\,dx=\int f(u)\,du$ metoda supstitucije

$$\int x\sin x^2dx=\begin{cases}u=x^2\\ du=2x\Rightarrow xdx=\frac{1}{2}du\end{cases}\Rightarrow\int x\sin x^2dx$$

$$\int\sin u\frac{1}{2}du=\frac{1}{2}\left(-\cos u\right)+C=-\frac{1}{2}\cos x^2+C\Leftrightarrow D_x\left(-\frac{1}{2}\cos x^2+C\right)=x\sin x^2$$

10. $\int u\;dv=uv-\int v\;du\Leftrightarrow\int f(x)g'(x)\,dx=f(x)g(x)-\int f'(x)g(x)\,dx$

$$\int\underbrace{\ln x}_{u}\;\underbrace{dx}_{dv}=\begin{cases}u=\ln x\rightarrow du=\frac{1}{x}dx\\ dv=dx\rightarrow v=x\end{cases}\Rightarrow\ln x\cdot x-\int x\cdot\frac{1}{x}dx$$

$$\ln x\cdot x-\int x\cdot\frac{1}{x}dx=x\ln x-x+C\Leftrightarrow D_x\left(x\ln x-x+C\right)=\ln x+\frac{x}{x}-1=\ln x$$

11. Metoda parcijalnih razlomaka - opsirnije objasnjeno u nastavku

Integrali poznatijih funkcija ili izraza:

$\int u^n du = \frac{u^{n+1}}{n+1} + C$

$\int \frac{du}{u} = \ln|u| + C$

$\int \sin u \; du = -\cos u + C$

$\int \cos u \; du = \sin u + C$

$\int \tan u \; du = \ln|\sec u| + C = -\ln|\cos u| + C$

$\int \cot u \; du = \ln|\sin u| + C$

$\int \sec^2 u \; du = \tan u + C$

$\int \csc^2 u \; du = -\cot u + C$

$\int \sec u \tan u \; du = \sec u + C$

$\int \csc u \cot u \; du = -\csc u + C$

$\int a^u du = \frac{a^u}{\ln a} + C; \; a > 0, a \neq 1$

$\int e^u du = e^u + C$

$\int \sinh u \; du = \cosh u + C$

$\int \cosh u \; du = \sinh u + C$

$\int \tanh u \; du = \ln \cosh u + C$

$\int \coth u \; du = \ln|\sinh u| + C$

$\int \operatorname{sech} u \; du = \tan^{-1}(\sinh u) + C$

$\int \operatorname{csch} u \; du = -\coth^{-1}(\cosh u) + C$

$\int \operatorname{sech}^2 u \; du = \tan u + C$

$\int \operatorname{csch}^2 u \; du = -\coth u + C$

$\int \operatorname{sech} u \; \tanh u \; du = -\operatorname{sech} u + C$

$\int \operatorname{csch} u \; \coth u \; du = -\operatorname{csch} u + C$

$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$

$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \frac{u-a}{u+a} + C$

$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = \frac{1}{a} \ln\left|\frac{u}{a + \sqrt{a^2 \pm u^2}}\right| + C$

$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{u} + C = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

$\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

$\int e^{au} \cos bu \; du = \frac{e^{au}(a \cos bu - b \sin bu)}{a^2 + b^2} + C$

$\int \frac{du}{\sqrt{a^2 \pm u^2}} = \sin^{-1}\frac{u}{a} + C = -\cos^{-1}\frac{u}{a} + C$

$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C = -\frac{1}{a}\cot^{-1}\frac{u}{a} + C$

$\int e^{au} \sin bu \; du = \frac{e^{au}(a \sin bu - b \cos bu)}{a^2 + b^2} + C$

$\int \csc u \; du = \ln|\csc u - \cot u| + C = \ln\left|\tan\frac{u}{2}\right| + C$

$\int \sec u \; du = \ln|\sec u + \tan u| + C = \ln\left|\tan\left(\frac{u}{2} + \frac{\pi}{4}\right)\right| + C$

$\int \sqrt{u^2 \pm a^2}\, du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$

7.2 Neodredjeni integral razlomljene racionalne funkcije

Integral razlomljene racionalne funkcije je naj rajrasireniji integral i pojavljuje se u vise razlicitih oblika, zavisno o stupnju potencije u brojniku i nazivniku izraza. Integrala ima opci oblik:

$I \equiv \int \frac{P(x)}{Q(x)}dx$ gdje su $P(x)$ i $Q(x)$ polinomi n-og stupnja.

Posto potencija polinoma u brojniku i nazivniku moze poprimati razlicite vrijednosti, pojavljuju se razlicite kombinacije razlomljene racionalne funkcije, koje su razmotrene u nastavku, svaka posebno.

Integral je oblika : $\mathrm{I} \equiv \int \frac{\mathrm{P_n}(\mathrm{x})}{\mathrm{Q_n}(\mathrm{x})}\mathrm{dx}$ potencija brojnika je veca od potencije nazivnika

1. $I \equiv \int \frac{3x^3-4x^2+3x}{x^2+1}dx = \int\left(3x-4+\frac{4}{x^2+1}\right)dx = \int 3xdx - \int 4dx + \int \frac{4}{x^2+1}dx$

$I \equiv \frac{3x^2}{2} - 4x + 4\tan^{-1}$ $\Rightarrow$ Podijelili smo razlomak. Postupak djeljenja prikazan je u srednjoskolskoj matematici u dijelu "Polinomi":

$\left(3x^3-4x^2+3x\right) \div \left(x^2+1\right) = 3x-4$ i ostatak 4 odnosno $\frac{4}{x^2+1}$

2. $I \equiv \int \frac{2x^3+7x^2+4x+2}{2x+3}dx = \int \frac{1}{2}\frac{2x^3+7x^2+4x+2}{x+\frac{3}{2}}dx = \frac{1}{2}\int\left(\frac{2x^3}{3}+\frac{4x^2}{2}-2x+\frac{5}{\left(x+\frac{3}{2}\right)}\right)dx$

$$I \equiv \frac{1}{2}\left[\int \frac{2x^3}{3}dx + \int \frac{4x^2}{2}dx - \int \frac{1}{2}(2x)dx + \int \frac{1}{2}\frac{5}{\left(x+\frac{3}{2}\right)}dx\right] = \underline{\frac{1}{2}\left(\frac{2x^3}{3}+2x^2-2x+5\ln\left|x+\frac{3}{2}\right|\right)+C}$$

Djeljenje daje rezultat: $\left(2x^3+7x^2+4x+2\right) \div \left(x+\frac{3}{2}\right) = \frac{2x^3}{3}+\frac{4x^2}{2}-2x+\frac{5}{x+\frac{3}{2}}$

Integral je oblika $\mathrm{I} \equiv \int \frac{1}{\mathrm{Q_2}(\mathrm{x})}\mathrm{dx}$ nazivnik je kvadratna funkcija

3. $I \equiv \int \frac{dx}{x^2+10x+30} = \int \frac{dx}{(x+5)^2+\left(\sqrt{5}\right)^2} \equiv \int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} = \underline{\frac{1}{\sqrt{5}}\tan^{-1}\frac{x+5}{\sqrt{5}}+C}$

Nadopunimo izraz na potpuni kvadrat i primijenimo poznati integral kao rjesenje:

$$x^2+10x+30=\underbrace{x^2+2x\cdot5+25}_{(x+5)^2}-25+30=(x+5)^2+5=(x+5)^2+\left(\sqrt{5}\right)^2$$

4. $$I\equiv\int\frac{dx}{3x^2-2x+5}=\frac{1}{3}\int\frac{dx}{\left(x-\frac{1}{3}\right)^2+\left(\frac{14}{\sqrt{9}}\right)^2}\equiv\frac{1}{a}\tan^{-1}\frac{u}{a}=\frac{1}{3}\frac{3}{\sqrt{14}}\tan^{-1}\frac{3\left(x-\frac{1}{3}\right)}{\sqrt{14}}=$$

$$\underline{I\equiv\frac{1}{\sqrt{14}}\tan^{-1}\left(\frac{3x-1}{\sqrt{14}}\right)+C}$$

$$3x^2-2x+5=\frac{1}{3}\left(x^2-\frac{2x}{3}+\frac{5}{3}\right)=\frac{1}{3}\left(\underbrace{x^2-2x\cdot\frac{1}{3}+\frac{1}{3^2}}_{\left(x-\frac{1}{3}\right)^2}-\frac{1}{9}+\frac{15}{9}\right)=\frac{1}{3}\left[\left(x-\frac{1}{3}\right)^2+\left(\sqrt{\frac{14}{9}}\right)^2\right]$$

5. $$I\equiv\int\frac{dx}{2x^2+2x+5}=\frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2}\equiv\frac{1}{a}\tan^{-1}\frac{u}{a}=\underline{\frac{1}{3}\tan^{-1}\left(\frac{2x+1}{3}\right)+C}$$

$$2x^2+2x+5=2\left(x^2+x+\frac{5}{2}\right)=2\left(\underbrace{x^2+2x\cdot\frac{1}{2}+\frac{1}{4}}_{\left(x+\frac{1}{2}\right)^2}-\frac{1}{4}+\frac{10}{4}\right)=2\left[\left(x+\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2\right]$$

6. $$I\equiv\int\frac{dx}{x^2-7x+12}=\int\frac{dx}{\left(x-\frac{7}{2}\right)^2-\left(\frac{1}{2}\right)^2}\equiv\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|=\frac{1}{2\cdot\frac{1}{2}}\ln\left|\frac{x-\frac{7}{2}-\frac{1}{2}}{x-\frac{7}{2}+\frac{1}{2}}\right|=\ln\left|\frac{x-4}{x-3}\right|+C$$

$$x^2-7x+12=\left(x^2-2x\frac{7}{2}+\frac{49}{4}-\frac{49}{4}+\frac{48}{4}\right)=\left(\underbrace{x^2-2x\frac{7}{2}+\frac{49}{4}}_{\left(x-\frac{7}{2}\right)^2}-\frac{1}{4}\right)=\left(x-\frac{7}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

7. $$I \equiv \int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\sqrt{\frac{3}{4}}\right)^2} \equiv \frac{1}{a}\tan^{-1}\frac{u}{a} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}} =$$

$$I \equiv \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$x^2+x+1 = \left(\underbrace{x^2+2x\cdot\frac{1}{2}+\frac{1}{4}}_{\left(x+\frac{1}{2}\right)^2} - \frac{1}{4}+\frac{4}{4}\right) = \left(x+\frac{1}{2}\right)^2 + \left(\sqrt{\frac{3}{4}}\right)^2$$

Integral je oblika $I \equiv \int \frac{1}{\sqrt{Q_2(x)}}dx$ nazivnik je kvadratna funkcija pod korjenom

8. $$I \equiv \int \frac{dx}{\sqrt{20-8x-x^2}} = \int \frac{dx}{\sqrt{6^2-(x-4)^2}} \equiv \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} = \sin^{-1}\frac{x-4}{6} + C$$

$$20-8x-x^2 = -\left(x^2-8x-20\right) = -\left(x^2-2x\cdot 4+16-16-20\right) = -\left[(x-4)^2-36\right] = 6^2-(x-4)^2$$

9. $$I \equiv \int \frac{dx}{\sqrt{1-x-3x^2}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(\frac{\sqrt{13}}{6}\right)^2 - \left(x+\frac{1}{6}\right)^2}} \equiv \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} = \frac{1}{\sqrt{3}}\sin^{-1}\frac{x+\frac{1}{6}}{\frac{\sqrt{13}}{6}}$$

$$I \equiv= \frac{1}{\sqrt{3}}\sin^{-1}\frac{6x+1}{\sqrt{13}} + C$$

$$1-x-3x^2 = -3\left(x^2+\frac{x}{3}-\frac{1}{3}\right) = -3\left(x^2+2x\cdot\frac{1}{6}+\frac{1}{36}-\frac{1}{36}-\frac{12}{36}\right) = -3\left[\left(x+\frac{1}{6}\right)^2-\frac{13}{36}\right]$$

$$= 3\left[\left(\frac{\sqrt{13}}{6}\right)^2 - \left(x+\frac{1}{6}\right)^2\right]$$

10. $$I \equiv \int \frac{dx}{\sqrt{28-12x-x^2}} = \int \frac{dx}{\sqrt{8^2-(x+6)^2}} \equiv \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} = \sin^{-1}\frac{x+6}{8} + C$$

$$28-12x-x^2 = -\left(x^2+12x-28\right) = -\left(x^2+2x\cdot 6+36-36-28\right) = -\left[(x+6)^2-64\right] =$$

$$= 8^2 - (x+6)^2$$

11. $$I \equiv \int \frac{dx}{\sqrt{1-3x+2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x-\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2}} \equiv \ln\left|u+\sqrt{u^2-a^2}\right| =$$

$$I \equiv \frac{1}{\sqrt{2}} \ln\left| x-\frac{3}{4}+\sqrt{\left(x-\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2} \right| + C$$

$$1-3x+2x^2 = 2\left(\frac{1}{2}-\frac{3}{2}x+x^2\right) = 2\left(x^2-2x\cdot\frac{3}{4}+\frac{9}{16}-\frac{9}{16}+\frac{8}{16}\right) = 2\left[\left(x-\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2\right]$$

12. $$I \equiv \int \frac{dx}{\sqrt{4x+5x^2}} = \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{\left(x+\frac{2}{5}\right)^2 - \left(\frac{2}{5}\right)^2}} \equiv \ln\left|u+\sqrt{u^2-a^2}\right| =$$

$$I \equiv \frac{1}{\sqrt{5}} \ln\left| x+\frac{2}{5}+\sqrt{\left(x+\frac{2}{5}\right)^2-\left(\frac{2}{5}\right)^2} \right| + C$$

$$4x+5x^2 = 5\left(x^2+\frac{4x}{5}\right) = 5\left(x^2-2x\cdot\frac{2}{5}+\frac{4}{25}-\frac{4}{25}\right) = 5\left[\left(x-\frac{2}{5}\right)^2-\left(\frac{2}{5}\right)^2\right]$$

13. $$I \equiv \int \frac{dx}{\sqrt{25-16x^2}} = \frac{1}{4} \int \frac{dx}{\sqrt{\left(\frac{5}{4}\right)^2 - x^2}} \equiv \int \frac{du}{\sqrt{a^2-u^2}} = \frac{1}{4}\sin^{-1}\frac{4x}{5}+C$$

$$25-16x^2 = 16\left(\frac{25}{16}-x^2\right) = 16\left[\left(\frac{5}{4}\right)^2-x^2\right]$$

14. $$I \equiv \int \frac{dx}{\sqrt{4x^2-4x+5}} = \frac{1}{2} \int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2 - 1^2}} \equiv \ln\left|u+\sqrt{u^2-a^2}\right| =$$

$$I \equiv \frac{1}{2} \ln\left| x-\frac{1}{2}+\sqrt{\left(x-\frac{1}{2}\right)^2-1^2} \right| + C$$

$$4x^2-4x+5 = 4\left(x^2-2x\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+\frac{5}{4}\right) = 4\left[\left(x-\frac{1}{2}\right)^2-1^2\right]$$

Integral je oblika $I \equiv \int \sqrt{ax^2 + bx + c}\, dx$ kvadratna funkcija pod korjenom

15. $I \equiv \int \sqrt{x^2 + x + 1}\, dx = \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}$

Izraz pod korjenom smo preuredili: $x^2 + x + 1 = x^2 + 2x\frac{1}{2} + \frac{1}{4} - \frac{1}{4} + \frac{4}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

$I \equiv \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \Rightarrow$ oznacimo: $x + \frac{1}{2} = u;\ dx = du;\ a^2 = \frac{3}{4} \Rightarrow a = \frac{\sqrt{3}}{2}$ pa mozemo pisati:

$$I \equiv \int \sqrt{u^2 + a^2}\, du \Rightarrow \int \sqrt{u^2 + a^2}\, du \frac{\sqrt{u^2 + a^2}}{\sqrt{u^2 + a^2}} = \int \frac{u^2 + a^2}{\sqrt{u^2 + a^2}} du = \underbrace{\int \frac{u^2}{\sqrt{u^2 + a^2}} du}_{I_1} +$$

$$+a^2 \underbrace{\int \frac{du}{\sqrt{u^2 + a^2}}}_{I_2} \Rightarrow I \equiv I_1 + a^2 I_2 \Rightarrow \text{rijesimo svaki integral posebno:}$$

$I_1 = \int \frac{u^2}{\sqrt{u^2 + a^2}} du = \int u \frac{u}{\sqrt{u^2 + a^2}} du \Rightarrow$ integral rijesimo metodom parcijalne integracije:

$$\begin{cases} t = u \\ du = dt \end{cases} \Rightarrow \begin{matrix} t^2 + a^2 = k \\ 2tdt = dk \end{matrix} \Leftrightarrow \int u \frac{u}{\sqrt{u^2 + a^2}} du \Downarrow v = \int \frac{tdt}{\sqrt{t^2 + a^2}} = \frac{1}{2}\int k^{-\frac{1}{2}} dk = \sqrt{k} = \sqrt{u^2 + a^2}$$

$$\int u \frac{u}{\sqrt{u^2 + a^2}} du = \underset{u}{t}\underbrace{\sqrt{t^2 + a^2}}_{v} - \int \underbrace{\sqrt{t^2 + a^2}}_{v}\, \underbrace{dt}_{du} = u\sqrt{u^2 + a^2} - \underbrace{\int \sqrt{u^2 + a^2}}_{I}$$

$I = u\sqrt{u^2 + a^2} - I + a^2 I_2 \Rightarrow I = \frac{1}{2} u\sqrt{u^2 + a^2} + \frac{a^2}{2} I_2$ rijesimo sada I_2 :

$I_2 = \int \frac{du}{\sqrt{u^2 + a^2}} = \ln\left|u + \sqrt{u^2 + a^2}\right| \to$ tipski integral. Sada mozemo napisati cijeli izraz:

$$I = \frac{1}{2} u\sqrt{u^2 + a^2} + \frac{a^2}{2} \ln\left|u + \sqrt{u^2 + a^2}\right| \text{ odnosno:}$$

$$I = \frac{1}{2}\left(x + \frac{1}{2}\right)\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{3}{8}\ln\left|\left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right| + C$$

$$I = \frac{1}{2}\left(x + \frac{1}{2}\right)\sqrt{x^2 + x + 1} + \frac{3}{8}\ln\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1}\right| + C$$

16. $I = \int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - x^2}\, dx \frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}} = \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx =$

$$I=a^2\underbrace{\int\frac{dx}{\sqrt{a^2-x^2}}}_{I_1}-\underbrace{\int\frac{x^2}{\sqrt{a^2-x^2}}dx}_{I_2}\Rightarrow I=a^2I_1-I_2\qquad I_1=\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+C$$

$$I_2=\int\frac{x^2}{\sqrt{a^2-x^2}}dx=\int x\frac{x}{\sqrt{a^2-x^2}}dx\Rightarrow\left\{\begin{matrix}u=x\\du=dx\end{matrix};dv=\frac{x}{\sqrt{a^2-x^2}};v=\int\frac{x}{\sqrt{a^2-x^2}}\right\}$$

$$v=\int\frac{x}{\sqrt{a^2-x^2}}\Rightarrow\left\{\begin{matrix}a^2-x^2=k\\-2xdx=dk\Rightarrow xdx=-\frac{1}{2}dk\end{matrix}\right\}\Rightarrow v=-\frac{1}{2}\int\frac{dk}{\sqrt{k}}=-\sqrt{k}=-\sqrt{a^2-x^2}$$

$$I_2=uv-\int vdu=x\left(-\sqrt{a^2-x^2}\right)-\underbrace{\left(-\int\frac{x}{\sqrt{a^2-x^2}}dx\right)}_{I}=-x\sqrt{a^2-x^2}+I=-x\sqrt{a^2-x^2}+I$$

$$I=a^2I_1-I_2=a^2\sin^{-1}\frac{x}{a}-\left(-x\sqrt{a^2-x^2}+I\right)\Rightarrow I=a^2\sin^{-1}\frac{x}{a}+x\sqrt{a^2-x^2}-I$$

$$2I=a^2\sin^{-1}\frac{x}{a}+x\sqrt{a^2-x^2}\Rightarrow \underline{I=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C}$$

Integral je oblika $I\equiv\int\frac{P(x)}{Q(x)}dx$ potencija polinoma brojnika i nazivnika je istog stupnja

17. $I=\int\frac{3x-2}{2x-5}dx=\frac{1}{2}\int\left(3+\frac{11}{2}\cdot\frac{1}{x-\frac{5}{2}}\right)dx=\frac{1}{2}\int 3dx+\frac{1}{2}\cdot\frac{11}{2}\int\frac{dx}{\left(x-\frac{5}{2}\right)}=\frac{3}{2}x+\frac{11}{4}\ln\left|x-\frac{5}{2}\right|+C$

Izraz $\frac{3x-2}{2x-5}$ smo preuredili: $\frac{3x-2}{2x-5}=\frac{1}{2}\cdot\frac{3x-2}{x-\frac{5}{2}}\Rightarrow(3x-2)\div\left(x-\frac{5}{2}\right)=3+\frac{11}{2}\cdot\frac{1}{\left(x-\frac{5}{2}\right)}$

Integral je oblika $I\equiv\int\frac{P(x)}{Q(x)}dx$ potencija polinoma nazivnika je veca od potencije polinoma brojnika

Metoda se svodi na preuredjenje nazivnika tako, da se prikaze u obliku produkta faktora i potom rijesi kao suma parcijalnih razlomaka:

18. $\int\frac{x}{\left(x^2+5x+6\right)^2}dx=\int\frac{x}{(x+2)^2(x+3)^2}dx=\int\left[\frac{A}{(x+2)^2}+\frac{B}{(x+2)}+\frac{C}{(x+3)^2}+\frac{D}{(x+3)}\right]dx$

Izraz smo preuredili: $\left(x^2+5x+6\right)^2 \Rightarrow \left[(x+2)(x+3)\right]^2 = (x+2)^2(x+3)^2$

Pisemo: $x \equiv A(x+3)^2 + B(x+2)(x+3)^2 + C(x+2)^2 + D(x+2)^2(x+3)$

Za vrijednosti korjena izraza, $x_1 = -2 \Rightarrow \underline{A=-2}$ $\quad x_2 = -3 \Rightarrow \underline{C=-3}$

Za $x = 0 \Rightarrow 0 \equiv A(x+3)^2 + B(x+2)(x+3)^2 + C(x+2)^2 + D(x+2)^2(x+3) =$

$$0 = (-2)(0+3)^2 + B(0+2)(0+3)^2 + (-3)(0+2)^2 + D(0+2)^2(0+3) =$$

$$0 = -2\cdot 9 + 18B - 12 + 12D \Rightarrow 18B + 12D = 30 \Rightarrow \underline{3B+2D=5}$$

Za $x = -4$

$$-4 = (-2)(-4+3)^2 + B(-4+2)(-4+3)^2 + (-3)(-4+2)^2 + D(-4+2)^2(-4+3)$$

$$-4 = -2\cdot 1 + B(-2)(1) + (-3)4 + D4(-1) \Rightarrow -2B - 4D = 10 \Rightarrow \underline{2B+4D=-10}$$

Rijesimo jednadzbe: $3B + 2D = 5 \quad 3\cdot 5 + 2D = 5 \Rightarrow 2D = -10 \Rightarrow \underline{D=-5}$

$2B + 4D = -10 \Rightarrow \underline{B+2D=-5} \Rightarrow 2B = 10 \Rightarrow \underline{B=5} \Uparrow$

$$I = \int\left[\frac{A}{(x+2)^2} + \frac{B}{(x+2)} + \frac{C}{(x+3)^2} + \frac{D}{(x+3)}\right]dx =$$

$$= \int\left[\frac{-2}{(x+2)^2} + \frac{5}{(x+2)} + \frac{-3}{(x+3)^2} + \frac{-5}{(x+3)}\right]dx =$$

$$= -2\int\frac{dx}{(x+2)^2} + 5\int\frac{dx}{(x+2)} - 3\int\frac{dx}{(x+3)^2} - 5\int\frac{dx}{(x+3)}$$

Uvedimo zamjenu za: $x+2 = u; dx = du \qquad x+3 = v; dx = dv$

$$I = -2\int u^{-2}du + 5\int\frac{du}{u} - 3\int v^{-2}dv - 5\int\frac{dv}{v} = \frac{-2}{-1}u^{-1} + 5\ln|u| - \frac{3}{-1}v^{-1} - 5\ln|v| + C$$

$$\underline{I = \frac{2}{(x+2)} + 5\ln|(x+2)| + \frac{3}{(x+3)} - 5\ln|(x+3)| + C}$$

19. $I = \int\frac{5x-3}{x\left(3x^2-6x-9\right)}dx = \frac{1}{3}\int\frac{5x-3}{x(x-3)(x+1)}dx$

Izraz smo preuredili: $\left(3x^2-6x-9\right) \Rightarrow \frac{1}{3}\left(x^2-2x-3\right) = \frac{1}{3}(x-3)(x+1)$

$$I = \frac{1}{3}\int\frac{5x-3}{x(x-3)(x+1)}dx = \frac{1}{3}\int\left[\frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+1)}\right]dx:$$

$\Rightarrow \frac{5x-3}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+1)}$ sredimo

$$5x - 3 = A(x-3)(x+1) + Bx(x+1) + Cx(x-3) =$$

$$= Ax^2 + Ax - 3Ax - 3A + Bx^2 + Bx + Cx^2 - 3Cx$$

$$0\cdot x^2+5x-3=x^2\left(A+B+C\right)+x\left(-2A+B-3C\right)-3A$$

$$x^2=0\Rightarrow \underline{A+B+C=0}\Rightarrow \quad 1+B+C=0\Rightarrow \underline{B=-C}$$

$$5x\Rightarrow \underline{5=-2A+B-3C}\Rightarrow \quad -2\cdot 1+B-3C=5\Rightarrow \underline{B-3C=7}$$

$$-3=-3A\Rightarrow A=1 \qquad B=1 \quad C=-2$$

$$I=\frac{1}{3}\int\left[\frac{A}{x}+\frac{B}{\left(x-3\right)}+\frac{C}{\left(x+1\right)}\right]dx=\frac{1}{3}\left[\int\frac{dx}{x}+\int\frac{dx}{\left(x-3\right)}-2\int\frac{dx}{\left(x+1\right)}\right]$$

$$I=\frac{1}{3}\left(\ln\left|x\right|+\ln\left|x-3\right|-2\ln\left|x+1\right|\right)+C=\ln\sqrt[3]{x}+\ln\sqrt[3]{\frac{x-3}{\left(x-1\right)^2}}+C$$

$$I=\ln\sqrt[3]{\frac{x\left(x-3\right)}{\left(x-1\right)^2}}+C$$

Koeficijente parcijalnih razlomaka moglo se rijesiti i na drugaciji nacin:

Uvrstimo vrijednosti korjena razlomka $x=0,\ x=3,\ x=-1$ u poznati nam izraz:

Za $x=0\Rightarrow 5\cdot 0-3=A\left(0-3\right)\left(0+1\right)+B\cdot 0\left(0+1\right)+C\cdot 0\left(0-3\right)\Rightarrow -3=-3A\Rightarrow A=1$

Za $x=3\Rightarrow 5\cdot 3-3=A\left(3-3\right)\left(3+1\right)+B\cdot 3\left(3+1\right)+C\cdot 3\left(3-3\right)\Rightarrow 12=12B\Rightarrow B=1$

Za $x=-1\Rightarrow 5\cdot\left(-1\right)-3=A\left(-1-3\right)\left(-1+1\right)+B\cdot\left(-1\right)\left(-1+1\right)+C\cdot\left(-1\right)\left(-1-3\right)\Rightarrow C=-2$

20. $I=\int\frac{dx}{x^2\left(x^2-1\right)}=\int\frac{dx}{x^2\left(x+1\right)\left(x-1\right)}$

Izrazimo parcijalne razlomke: $1\equiv\frac{A}{x^2}+\frac{B}{x}+\frac{C}{\left(x+1\right)}+\frac{D}{\left(x-1\right)}$ i dalje, sredimo:

$1\equiv A\left(x^2-1\right)+Bx\left(x^2-1\right)+Cx^2\left(x-1\right)+Dx^2\left(x+1\right)$

$1\equiv Ax^2-A+Bx^3-Bx+Cx^3-Cx^2+Dx^3+Dx^2$

$1\equiv x^3\left(B+C+D\right)+x^2\left(A-C+D\right)+x\left(-B\right)-A$

Rijesimo sistem jednadzbi: $B=0\quad 1=-A\Rightarrow A=-1$

$B+C+D=0\Rightarrow 0+C+D=0\Rightarrow C+D=0\Rightarrow C=-\frac{1}{2}$

$A-C+D\Rightarrow -1-C+D=0\Rightarrow -C+D=1\Rightarrow D=\frac{1}{2}$

$$I=\int\frac{dx}{x^2\left(x^2-1\right)}=\int\frac{dx}{x^2\left(x+1\right)\left(x-1\right)}=\int\left[\frac{-dx}{x^2}+\overset{B=0}{\overset{\sqcap}{0}}-\frac{1}{2}\frac{dx}{\left(x+1\right)}+\frac{1}{2}\frac{dx}{\left(x-1\right)}\right]$$

$$I=-\int x^{-2}dx-\frac{1}{2}\int\frac{dx}{\left(x+1\right)}+\frac{1}{2}\int\frac{dx}{\left(x-1\right)}=\frac{1}{x}-\frac{1}{2}\ln\left|x+1\right|+\frac{1}{2}\ln\left|x-1\right|+C$$

$$I=\frac{1}{x}+\frac{1}{2}\left(\ln\left|x-1\right|-\ln\left|x+1\right|\right)+C=\frac{1}{x}+\ln\sqrt{\frac{x-1}{x+1}}+C=\ln\sqrt[3]{x}+\ln\sqrt[3]{\frac{x-3}{\left(x-1\right)^2}}+C$$

$$\underline{I=\frac{1}{x}+\ln\sqrt{\frac{x-1}{x+1}}+C}$$

21. $I=\int\frac{dx}{x\left(x^2+2x+3\right)}$

Izraz u nazivniku ima konjugirano kompleksne korjene i postupak rjesavanja je drugaciji.

Parcijalni razlomak mora imati oblik $\frac{Ax+B}{x+i}$:

$\frac{1}{x\left(x^2+2x+3\right)}\equiv\frac{A}{x}+\frac{Bx+C}{x^2+2x+3}$ i dalje, sredimo:$1\equiv A\left(x^2+2x+3\right)+\left(Bx+C\right)x$

$1\equiv Ax^2+2Ax+3A+Bx^2+Cx=x^2\left(A+B\right)+x\left(2A+C\right)+3A$

Rijesimo sistem jednadzbi: $1=3A\Rightarrow A=\frac{1}{3}$

$A+B=0\Rightarrow\frac{1}{3}+B=0\Rightarrow B=-\frac{1}{3}\Rightarrow 2A+C=2\frac{1}{3}+C=0\Rightarrow C=-\frac{2}{3}$

$$I=\int\frac{dx}{x\left(x^2+2x+3\right)}=\int\left[\frac{1}{3x}+\frac{\left(-\frac{1}{3}\right)x+\left(-\frac{2}{3}\right)}{\left(x^2+2x+3\right)}dx\right]=\frac{1}{3}\int\left(\frac{1}{x}-\frac{x+2}{\left(x^2+2x+3\right)}\right)dx$$

$$I=\frac{1}{3}\left[\int\frac{dx}{x}-\underbrace{\int\frac{x+2}{\left(x^2+2x+3\right)}dx}_{I_1}\right]=\frac{1}{3}\left[\ln|x|-I_1\right]\Rightarrow \underline{I=\frac{1}{3}\left[\ln|x|-I_1\right]}$$ rijesimo sada I_1 :

$$I_1=\int\frac{x+2}{\left(x^2+2x+3\right)}dx=\underbrace{\int\frac{x}{\left(x^2+2x+3\right)}dx}_{I_2}+2\underbrace{\int\frac{dx}{\left(x^2+2x+3\right)}}_{I_3}\Rightarrow I_1=I_2+2I_3$$

$$I_2=\int\frac{x}{\left(x^2+2x+3\right)}dx\Rightarrow\left\{\begin{array}{c}x^2+2x+3=k\\(2x+2)\,dx=dk\\x=\frac{2x+2}{2}-1\end{array}\right\}\Rightarrow\int\frac{\frac{2x+2}{2}-1}{x^2+2x+3}dx$$

$$I_2=\frac{1}{2}\int\frac{2x-2}{x^2+2x+3}dx-\int\frac{dx}{x^2+2x+3}+C=\frac{1}{2}\int\frac{dk}{k}-\underbrace{\int\frac{dx}{x^2+2x+3}}_{I_3}+C$$

$$I_2==\frac{1}{2}\ln|k|-I_3=\frac{1}{2}\ln\left|x^2+2x+3\right|-I_3$$

$$I=\frac{1}{3}\left[\ln|x|-I_1\right]=\frac{1}{3}\left[\ln|x|-I_3+2I_3\right]=\frac{1}{3}\left[\ln|x|-\frac{1}{2}\ln\left|x^2+2x+3\right|+I_3\right]$$

$$I_3=\int\frac{dx}{x^2+2x+3}\Rightarrow\left\{\begin{array}{c}x^2+2x+3=x^2+2x+1-1+3=(x+1)^2+\left(\sqrt{2}\right)^2\\x+1=u;dx=du;a=\sqrt{2};a^2=2\end{array}\right\}=$$

$$I_3=\int\frac{du}{u^2+a^2}=\frac{1}{a}\tan^{-1}\frac{u}{a}$$

$$I=\frac{1}{3}\left[\ln|x|-\frac{1}{2}\ln\left|x^2+2x+3\right|+\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}\right]+C$$

$$\underline{I=\frac{1}{3}\left[\ln\frac{x}{\sqrt{x^2+2x+3}}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}\right]+C}$$

Integral je oblika $I \equiv \int x, \sqrt{\frac{P(x)}{Q(x)}} dx$ kombinacija funkcije polinoma u prvoj potenciji pod korjenom

22. $$I = \int \frac{\sqrt{3x-1}}{x^2} dx \Rightarrow \left\{ \begin{array}{l} \sqrt{3x-1} = u \ \ /^2 \Rightarrow 3x-1 = u^2 \\ x = \frac{u^2+1}{3} \Rightarrow x^2 = \frac{\left(u^2+1\right)^2}{9};\ dx = \frac{2}{3} u du \end{array} \right\} \Rightarrow$$

$$I = \int \frac{2}{3} \frac{9u^2}{\left(u^2+1\right)^2} du = 6 \int \frac{u^2}{\left(u^2+1\right)^2} du = 6 I_1$$

$$I_1 = \int \frac{u^2}{\left(u^2+1\right)^2} du \Rightarrow \frac{u^2}{\left(u^2+1\right)^2} \frac{u^2}{\left(u+1\right)^2 \left(u-1\right)^2} = \frac{A}{\left(u+1\right)^2} + \frac{B}{\left(u+1\right)} + \frac{C}{\left(u-1\right)^2} + \frac{D}{\left(u-1\right)}$$

Za $u = 1 \Rightarrow 1^2 = A0 + B0 + 4C + D0 \Rightarrow 4C = 1 \Rightarrow C = \frac{1}{4}$

Za $u = -1 \Rightarrow \left(-1\right)^2 = 4A + B0 + C0 + D0 \Rightarrow 4A = 1 \Rightarrow A = \frac{1}{4}$

Za $u = 0$ $\quad \left(0\right)^2 = A + B + C - D \Rightarrow \frac{1}{4} + B + \frac{1}{4} - D = 0 \Rightarrow B - D = -\frac{1}{2}$

Za $u = 2$ $\quad \left(2\right)^2 = A + 3B + 9C + 9D = 4 = \frac{1}{4} + 3B + \frac{9}{4} + 9D \Rightarrow B + 3D = \frac{1}{2}$

Koeficijenti imaju vrijednosti: $A = \frac{1}{4} \quad B = -\frac{1}{4} \quad C = \frac{1}{4} \quad D = \frac{1}{4}$

$$I_1 = \frac{1}{4} \int \left[\frac{1}{\left(u+1\right)^2} - \frac{1}{\left(u+1\right)} + \frac{1}{\left(u-1\right)^2} + \frac{1}{\left(u-1\right)} \right] du$$

$$I_1 = \frac{1}{4} \left[\int \frac{du}{\left(u+1\right)^2} - \int \frac{du}{\left(u+1\right)} + \int \frac{du}{\left(u-1\right)^2} + \int \frac{du}{\left(u-1\right)} \right] \Leftrightarrow I = 6 I_1$$

$$I = 6 I_1 = \frac{6}{4} \left[-\frac{1}{u+1} - \ln\left|u+1\right| - \frac{1}{u-1} + \ln\left|u-1\right| \right] = \underline{\frac{3}{2} \ln \frac{\sqrt{3x-1}-1}{\sqrt{3x-1}+1} - \frac{\sqrt{3x-1}}{x} + C}$$

23. $$I = \int 2x \sqrt[3]{x-2}\, dx \Rightarrow \left\{ \begin{array}{l} \sqrt[3]{x-2} = u \Rightarrow x - 2 = u^3 \\ x = u^3 + 2 \Rightarrow dx = 3u^2 du \end{array} \right\} \Rightarrow \int 2\left(u^3+2\right) u 3u^2 du$$

$$I = 6 \int u^3 \left(u^3 + 2\right) du = 6 \int u^6 du + 12 \int u^3 du = \frac{6}{7} u^7 + \frac{12}{4} u^4$$

$$I = \underline{\frac{6}{7} \sqrt[3]{\left(x-2\right)^7} + 3 \sqrt[3]{\left(x-2\right)^4} + C}$$

Rjesavanje integrala primjenom ' brze formule ' (tocka 8. pravila integriranja)

$$\int [f(x)]^n f'(x)dx = \frac{1}{n+1}[f(x)]^{n+1} + C \text{ za sve racionalne brojeve; } n \neq -1$$

24. $I = \int (2x+3)^2\, dx = \int (2x+3)^2 \underbrace{2\frac{1}{2}}_{(2x+3)'}\, dx = \frac{1}{2}\int (2x+3)^2\, 2dx = \frac{1}{2}\frac{(2x+3)^3}{3} = \underline{\frac{1}{6}(2x+3)^3 + C}$

25. $I = \int (x^3-2)^2\, dx = \int (x^3-2)^2\, 3x^2 dx = \underline{\frac{1}{3}(x^3-2)^3 + C}$

26. $I = \int \frac{5x^2}{(x^3-2)^3}\, dx = 5\int \frac{3x^2}{(x^3-2)^3}\frac{1}{3}dx = \frac{5}{3}\int (x^3-2)^{-3}\, 3x^2 dx = \frac{5}{3}\cdot\frac{1}{(-2)}(x^3-2)^{-2}$

$I = \underline{-\frac{5}{6}(x^3-2)^{-2} + C}$

27. $I = \int \sin^2 x \cos x\, dx = \int (\sin x)^2 \overbrace{\cos x}^{(\sin x)'}\, dx = \frac{(\sin x)^3}{3} + C = \underline{\frac{1}{3}\sin^3 x + C}$

28. $I = \int 3x\sqrt{1-2x^2}\, dx = 3\int x\sqrt{1-2x^2}\,(-4)\left(-\frac{1}{4}\right) dx = -\frac{3}{4}\int -4x(1-2x^2)^{\frac{1}{2}}\, dx$

$I = -\frac{3}{4}\frac{1}{\frac{3}{2}}(1-2x^2)^{\frac{3}{2}} = \underline{-\frac{1}{2}\sqrt{(1-2x^2)^3} + C}$

29. $I = \int \frac{x^2}{\sqrt[4]{x^3+1}}\, dx = \int 3x^2(x^3+1)^{-\frac{1}{4}}\frac{1}{3}dx = \frac{1}{3}\cdot\frac{1}{\frac{3}{4}}(x^3+1)^{\frac{3}{4}} = \underline{\frac{4}{9}\sqrt[4]{(x^3+1)^3} + C}$

30. $I = \int \frac{x}{(x^2+3)^2}\, dx = \frac{1}{4}\int 4x(x^2+3)^{-2}\, dx = \frac{1}{4}\cdot\frac{1}{-1}(x^2+3)^{-1} = \underline{-\frac{1}{4(x^2+3)} + C}$

7.3 Neodredjeni integral trigonometrijskih funkcija

Integral je oblika $I=\int \sin x, \cos x \, dx$

Integrali ovog tipa rjesavaju se supstitucijom, uvodjenjem nove promjenjive i to:

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2u}{1+u^2} \Rightarrow \sin x = \frac{2u}{1+u^2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \left(\cos^2\frac{x}{2} - \cos^2\frac{x}{2}\cdot\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right) = \cos^2\frac{x}{2}\left(1-\tan^2\frac{x}{2}\right) = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\cos x = \frac{1-u^2}{1+u^2}$$

$$\tan\frac{x}{2} = u \Rightarrow \frac{x}{2} = \tan^{-1} u \Rightarrow x = 2\tan^{-1} u \Rightarrow dx = 2\left(\tan^{-1} u\right)' = \frac{2du}{1+u^2} \Rightarrow dx = \frac{2du}{1+u^2}$$

31. $$I = \int \frac{dx}{5-3\cos x} = \int \frac{\frac{2du}{1+u^2}}{5-3\left(\frac{1-u^2}{1+u^2}\right)} = \int \frac{\frac{2}{1+u^2}}{\frac{5\left(1+u^2\right)-3\left(1-u^2\right)}{1+u^2}} du = \int \frac{2}{5+5u^2-3+3u^2} du$$

$$I = \int \frac{2}{2+8u^2} du = \int \frac{du}{1-\left(2u\right)^2} \Rightarrow \left\{\begin{matrix} 2u = k \\ 2du = dk \Rightarrow du = \frac{dk}{2} \end{matrix}\right\} \Rightarrow \frac{1}{2}\int \frac{dk}{1-k^2} = \frac{1}{2}\frac{1}{a}\tan^{-1}\frac{k}{a}$$

$$I = \frac{1}{2}\tan^{-1} k \Rightarrow \frac{1}{2}\tan^{-1} 2u = \frac{1}{2}\tan^{-1}\left(2\tan\frac{x}{2}\right) + C = \underline{\frac{1}{2}\tan^{-1}\left(2\tan\frac{x}{2}\right) + C}$$

32. $$I = \int \frac{\cos x}{1+\cos x} dx = \int \frac{\frac{1-u^2}{1+u^2}}{1+\frac{1-u^2}{1+u^2}}\frac{2du}{1+u^2} = \int \frac{\frac{1-u^2}{1+u^2}}{\frac{1+u^2+1-u^2}{1+u^2}}\frac{2du}{1+u^2} = \int \frac{1-u^2}{2}\frac{2du}{1+u^2}du$$

$$I = \int \frac{1-u^2}{1+u^2} du = -\int \frac{u^2-1}{1+u^2} du = -\int\left(1-\frac{1}{1+u^2}\right) du = -u + 2\tan^{-1} u + C$$

Izraz pod znakom integrala smo podijelili: $\left(u^2-1\right) \div \left(1+u^2\right) = 1-\frac{2}{1+u^2}$

$$I = -\tan\frac{x}{2} + 2\tan^{-1}\left(\tan\frac{x}{2}\right) + C = \underline{-\tan\frac{x}{2} + 2\tan^{-1}\left(\tan\frac{x}{2}\right) + C}$$

Integral je oblika $I=\int \text{sinhx}, \text{coshx}\, dx$

Integrali ovog tipa rjesavaju se supstitucijom uvodjenjem nove promjenjive:

$$\sinh x = 2\sinh\frac{x}{2}\cosh\frac{x}{2} = 2\tanh\frac{x}{2}\cosh^2\frac{x}{2} = \frac{2\tan\frac{x}{2}}{1-\tanh^2\frac{x}{2}} = \frac{2u}{1-u^2} \Rightarrow \sinh x = \frac{2u}{1-u^2}$$

$$\cosh x = \cosh^2\frac{x}{2} + \sinh^2\frac{x}{2} = \left(\cosh^2\frac{x}{2} + \cosh^2\frac{x}{2}\cdot\frac{\sinh^2\frac{x}{2}}{\cosh^2\frac{x}{2}}\right) = \cos^2\frac{x}{2}\left(1+\tan^2\frac{x}{2}\right) =$$

$$\cosh x = \frac{1+\tanh^2\frac{x}{2}}{1-\tanh^2\frac{x}{2}} \Rightarrow \cosh x = \frac{1+u^2}{1-u^2}$$

$$\tanh\frac{x}{2} = u \Rightarrow \frac{x}{2} = \tanh^{-1}u \Rightarrow x = 2\tanh^{-1}u \Rightarrow dx = 2\left(\tanh^{-1}u\right)' = \frac{2du}{1-u^2} \Rightarrow dx = \frac{2du}{1-u^2}$$

33. $$I=\int\frac{dx}{\cosh x} = \int\frac{\frac{2du}{1-u^2}}{\left(\frac{1+u^2}{1-u^2}\right)} = \int\frac{2\left(1-u^2\right)}{\left(1-u^2\right)\left(1+u^2\right)}du = 2\int\frac{1}{\left(1+u^2\right)}du = 2\tan^{-1}u$$

$$\underline{I = 2\tan^{-1}\left(\tanh\frac{x}{2}\right)+C}$$

Zadaci se mogu rijesiti supstitucijom koristenjem definicije hiperbolne funkcije:

34. $$I=\int\frac{dx}{\sinh x} \Rightarrow \left\{\begin{matrix}\sinh = \frac{e^x-e^{-x}}{2} \Rightarrow e^x = u; e^{-x} = \frac{1}{u} \\ e^x dx = du \Rightarrow dx = \frac{du}{e^x} \Rightarrow dx = \frac{du}{u}\end{matrix}\right\} \Rightarrow 2\int\frac{\frac{du}{u}}{e^x - e^{-x}} = 2\int\frac{du}{u\left(u-\frac{1}{u}\right)}$$

$$I = 2\int\frac{du}{u\left(\frac{u^2-1}{u}\right)} = 2\int\frac{du}{u^2-1} = 2\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = \underline{\ln\left|\frac{e^x-1}{e^x+1}\right| + C}$$

Integral je oblika $I=\int \sin mx \times \cos nx\, dx$

35. $I=\int \sin 5x \cos 9x dx \Rightarrow \left\{ \begin{array}{c} \text{Iz poznatih odnosa trigonometrijskih funkcija imamo:} \\ \sin\alpha \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right] \end{array} \right\}$

$$I=\frac{1}{2}\int\left[\sin(5x+9x)+\sin(5x-9x)\right]dx=\frac{1}{2}\int\left[\sin 14x+\sin(-4x)\right]dx=$$

$$I=\frac{1}{2}\int \sin 14x dx+\frac{1}{2}\int \sin(-4x)\,dx \quad \text{Supstitucija: } 14x=u; 4x=v; \cos(-x)=\cos x$$

$$I=\frac{1}{2}\cdot\frac{1}{14}(-\cos u)+\frac{1}{2}\cdot\frac{1}{4}(\cos v)=-\frac{1}{28}\cos 14x+\frac{1}{8}\cos 4x$$

$$\underline{I=-\frac{1}{28}\cos 14x+\frac{1}{8}\cos 4x}$$

36. $I=\int \sin(5x-1)\cdot \sin 3x dx \Rightarrow \left\{\sin\alpha\cdot\sin\beta=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]\right\}\Rightarrow$

$$I=\frac{1}{2}\int\left[\cos(5x-1-3x)-\cos(5x-1+3x)\right]=\frac{1}{2}\int\cos(2x-1)\,dx-\frac{1}{2}\int\cos(8x-1)dx$$

Rijesimo supstitucijom: $(2x-1)=u\rightarrow dx=\frac{du}{2};(8x-1)=v\rightarrow dx=\frac{du}{8}$

$$I=\frac{1}{2}\int\cos(u)\,dx-\frac{1}{2}\int\cos(v)dx=\frac{1}{2}\cdot\frac{1}{2}\sin u-\frac{1}{2}\frac{1}{8}\sin v=\frac{1}{4}\sin(2x-1)-\frac{1}{16}\sin(8x-1)+C$$

$$\underline{I=\frac{1}{4}\sin(2x-1)-\frac{1}{16}\sin(8x-1)+C}$$

Integral je oblika $I=\int \sin^n x;\cos^n x;\tan^n x;\cot^n x\, dx$

37. $I_*=\int \sin^2 x dx \Rightarrow \left\{ \begin{array}{c} \text{Rijesimo metodom parcijalne integracije } \int u dv = uv-\int v du \\ \sin^2 x=\underbrace{\sin x}_{u}\cdot\underbrace{\sin x dx}_{dv}\Rightarrow du=\cos x dx \Leftrightarrow v=\int \sin x dx=-\cos x \end{array} \right\}$

$$I_*=-\sin x\cos x-\int(-\cos x)\cos x dx=-\sin x\cos x+\int\cos^2 x dx$$

$$I_*=-\sin x\cos x+\int\left(1-\sin^2 x\right)dx=-\sin x\cos x+\int 1dx-\underbrace{\int\sin^2 x d\,x}_{I_*}$$

$$I_*=-\sin x\cos x+x-I_*\Rightarrow 2I_*=-\sin x\cos x+x\Rightarrow I_*=\frac{1}{2}(-\sin x\cos x+x)$$

$\underline{I_* = \frac{x}{2} - \frac{1}{2}\sin x \cos x + C}$ Mozemo pisati slijedecu formulu za rjesenje za bilo koji *n*:

$$I = \int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

$$I = \int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

38. $I = \int \tan^3 x dx \Rightarrow \left\{\begin{matrix} \text{Supstitucija: } \tan x = u \to x = \tan^{-1} u \\ dx = \dfrac{1}{1+u^2} du \end{matrix}\right\} \Rightarrow \int u^3 \frac{1}{1+u^2} du =$

$$I = \int \left(u - \frac{u}{u^2+1} \right) du = \int u du - \int \frac{u}{u^2+1} du = \frac{u^2}{2} - \int \frac{u}{u^2+1} du = \frac{u^2}{2} - \frac{1}{2}\underbrace{\int \frac{dk}{k}}_{u^2+1=k}$$

$$I = \frac{u^2}{2} - \frac{1}{2}\ln k = \frac{u^2}{2} - \frac{1}{2}\ln\left(u^2+1\right) = \frac{\tan^2 x}{2} - \frac{1}{2}\ln\left(\tan^2 x + 1\right) + C$$

$$\underline{I = \frac{1}{2}\tan^2 x - \frac{1}{2}\ln\left(\tan^2 x + 1\right) + C}$$

7.4 Razni zadaci

39. $I = \int x^3 e^{x^2} dx = \left\{\begin{matrix} u = x^2 & & dv = xe^{x^2} dx \\ du = 2x & v = \int xe^{x^2} dx = \left\{\begin{matrix} k = x^2 \\ dk = 2xdx \end{matrix}\right\} = \dfrac{1}{2}\int ke^k dk = \dfrac{1}{2}e^k & = \dfrac{1}{2}e^x \end{matrix}\right\}$

$$I = \int x^3 e^{x^2} dx = \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2} dx = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} = \underline{\frac{1}{2}e^{x^2}\left(x^2-1\right) + C}$$

40. $I = \int \frac{dx}{x^2-4} = \left\{\begin{matrix} \dfrac{1}{x^2-4} = \dfrac{1}{(x-2)(x+2)} = \dfrac{A}{(x-2)} + \dfrac{B}{(x+2)} \\ \underline{1 = A(x+2) + B(x-2)} \end{matrix}\right\} =$

$$I = \left\{\begin{matrix} x = -2 \to 1 = A(-2+2) + B(-2-2) \to B = -\dfrac{1}{4} \\ x = 2 \to 1 = A(2+2) + B(2-2) \to A = \dfrac{1}{4} \end{matrix}\right\} =$$

$$I = \int \frac{dx}{x^2-4} = \int \left(\frac{\frac{1}{4}}{(x-2)} + \frac{-\frac{1}{4}}{(x+2)} \right) dx = \frac{1}{4}\left(\int \frac{dx}{x-2} - \int \frac{dx}{x+2} \right) = \frac{1}{4}\left(\ln|x-2| - \ln|x+2| \right)$$

$$I=\frac{1}{4}\ln\frac{|x-2|}{|x+2|}+C$$

41. $I=\int\frac{dx}{x\sqrt{1-x}}\Rightarrow\left\{\begin{matrix}1-x=u^2\to x=1-u^2\\ dx=-2udu\end{matrix}\right\}\Rightarrow\int\frac{-2udu}{\left(1-u^2\right)u}=-2\int\frac{du}{1-u^2}$

$I=-2\int\frac{du}{1-u^2}=-2\int\frac{du}{(1-u)(1+u)}=-\ln\frac{|1+u|}{|1-u|}+C\Leftrightarrow$ Uvrstimo u osnovni integral:

$$I=\int\frac{dx}{x\sqrt{1-x}}=-\ln\frac{\left|1+\sqrt{1-x}\right|}{\left|1-\sqrt{1-x}\right|}=\ln\frac{\left|1-\sqrt{1-x}\right|}{\left|1+\sqrt{1-x}\right|}+C$$

42. $I=\int\sin^3 x\cos^2 xdx=\int\sin^2 x\cos^2 x\sin xdx=\int\left(1-\cos^2 x\right)\cos^2 x\sin xdx\Rightarrow$

$I\Rightarrow\left\{\begin{matrix}u=\cos x\\ du=-\sin xdx\end{matrix}\right\}=\int\left(1-u^2\right)u^2(-du)=-\int\left(u^2-u^4\right)du=\int u^4du-\int u^2du$

$$I=\frac{u^5}{5}-\frac{u^3}{3}=\frac{\cos x^5}{5}-\frac{\cos x^3}{3}+C$$

43. $I=\int\frac{x+1}{x^3+x^2-6x}dx\Rightarrow\left\{\begin{matrix}\frac{x+1}{x^3+x^2-6x}=\frac{x+1}{x\left(x^2+x-6\right)}=\frac{A}{x}+\frac{B}{(x-2)}\frac{C}{(x+3)}\\ x+1=A(x-2)(x+3)+Bx(x+3)+Cx(x-2)\end{matrix}\right\}=$

$$=\left\{\begin{matrix}\text{za } x=0:\ 0+1=A(-2)(3)+B(0)(3)+C(0)(-2)=-6A\to A=-\frac{1}{6}\\ \text{za } x=2:\ 2+1=A(0)(5)+B(2)(5)+C(2)(0)=10B\to B=\frac{3}{10}\\ \text{za } x=-3:\ -3+1=A(-5)(0)+B(-3)(0)+C(-3)(-5)=15C\to C=-\frac{2}{15}\end{matrix}\right\}\Rightarrow$$

$$I=\int\frac{x+1}{x^3+x^2-6x}dx=\int\left(\frac{A}{x}+\frac{B}{(x-2)}\frac{C}{(x+3)}\right)dx=\int\left(-\frac{1}{6}\frac{1}{x}+\frac{3}{10}\frac{1}{(x-2)}-\frac{2}{15}\frac{1}{(x+3)}\right)dx$$

$$I=-\frac{1}{6}\ln|x|+\frac{3}{10}\ln|x-2|-\frac{2}{15}\ln|x+3|+C$$

44. $I=\int\ln\left(x^2+2\right)dx\Rightarrow\left\{\begin{matrix}u=\ln\left(x^2+2\right) & dv=dx\\ du=\frac{2x}{\left(x^2+2\right)}dx & v=x\end{matrix}\right\}$

$$\int \ln\left(x^2+2\right)dx = x\ln\left(x^2+2\right) - \int x\frac{2x}{\left(x^2+2\right)}dx = x\ln\left(x^2+2\right) - 2\underbrace{\int\frac{x^2}{\left(x^2+2\right)}dx}_{x^2:\left(x^2+2\right)=1-\frac{2}{x^2+2}} =$$

$$I = x\ln\left(x^2+2\right) - 2\int\left(1-\frac{2}{x^2+2}\right)dx = x\ln\left(x^2+2\right) - 2\int dx + 4\int\frac{dx}{x^2+2} x\ln\left(x^2+2\right) - 2x +$$

$$+4\left\{\int\frac{dx}{x^2+2} = \int\frac{dx}{x^2+\left(\sqrt{2}\right)^2} \Rightarrow \int\frac{du}{u^2+a} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) \Rightarrow \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right\}$$

$$\underline{I = x\ln\left(x^2+2\right) - 2x + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C}$$

45. $$I = \int\frac{dx}{(x-2)\sqrt{x+2}} \Rightarrow \left\{\begin{matrix} x+2=u^2 \quad x=u^2-2 \\ dx = 2udu \end{matrix}\right\} \Rightarrow \int\frac{2udu}{\left(u^2-2-2\right)\sqrt{u^2-2+2}}$$

$$I = 2\int\frac{du}{\left(u^2-4\right)} = 2\left\{\int\frac{du}{\left(u^2-a^2\right)} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|\right\} = 2\frac{1}{4}\ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right|$$

$$\underline{I = \frac{1}{2}\ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right| + C}$$

46. $$I = \int\sin^4 x\cos^7 xdx = \int\sin^4 x\cos^6 x\cos xdx = \int\left(1-\sin^2 x\right)^3\sin^4 x\cos xdx \Rightarrow \left\{\begin{matrix} u = \sin x \\ du = \cos x\end{matrix}\right\}$$

$$I = \int\left(1-u^2\right)^3 u^4du = \int u^4\left(1-3u^2+3u^4-u^6\right)du = \int\left(u^5-3u^6+3u^8-u^{10}\right)du$$

$$I = \frac{1}{6}u^6 - \frac{3}{7}u^7 + \frac{3}{9}u^9 - \frac{1}{11}u^{11} = \underline{\frac{1}{6}\sin^6 x - \frac{3}{7}\sin^7 x + \frac{1}{3}\sin^9 x - \frac{1}{11}\sin^{11} x + C}$$

47. $$I = \int\frac{3x+5}{x^3-x^2-x+1}dx \Rightarrow \left\{\begin{matrix}\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \\ 3x+5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)\end{matrix}\right\} =$$

$$= \left\{\begin{matrix}\text{za } x=1:\ 3+5 = A(0)+B(2)(0)+C(2) = 2C \to C = 4 \\ \text{za } x=-1:\ -3+5 = A(4)+B(0)(-2)+C(0) = 4A \to A = \frac{1}{2} \\ \text{usporedimo koef. uz } x^2 \text{ na obje strane: } 0 = A+B \to B = -A = -\frac{1}{2}\end{matrix}\right\} \Rightarrow$$

$$I=\int\frac{3x+5}{x^3-x^2-x+1}dx=\int\left(\frac{1}{2(x+1)}-\frac{1}{2(x-1)}+\frac{4}{(x-1)^2}\right)dx=$$

$$I=\frac{1}{2}\ln|x+1|-\frac{1}{2}\ln|x-1|+\left\{4\int\frac{dx}{(x-1)^2}=4\int(x-1)^{-2}dx=4\frac{1}{-1}(x-1)^{-1}\right\}$$

$$\underline{I=\frac{1}{2}\ln|x+1|-\frac{1}{2}\ln|x-1|-\frac{4}{x-1}+C}$$

48. $I=\int x^2\ln x dx\Rightarrow\left\{\begin{matrix}u=\ln x. & dv=x^2dx\\ du=\frac{1}{x}dx & v=\int x^2dx=\frac{x^3}{3}\end{matrix}\right\}$

$$I=\int x^2\ln x dx=\underbrace{\ln x}_{u}\underbrace{\frac{x^3}{3}}_{v}-\int\underbrace{\frac{x^3}{3}}_{v}\underbrace{\frac{1}{x}dx}_{du}=\frac{x^3}{3}\ln x-\frac{1}{3}\int x^2dx=\frac{x^3}{3}\ln x-\frac{1}{3}\frac{x^3}{3}=$$

$$\underline{I=\frac{x^3}{3}\ln x-\frac{x^3}{9}+C}$$

49. $I=\int\sin^5 x dx=\int\left(\sin^2 x\right)^2\sin x dx=\int\left(1-\cos^2 x\right)^2\sin x dx\Rightarrow\left\{\begin{matrix}u=\cos x\\ du=-\sin x dx\end{matrix}\right\}$

$$I=-\int\left(1-u^2\right)^2du=-\int\left(1-2u^2+u^4\right)du=-\left(u-\frac{2}{3}u^3+\frac{1}{5}u^5\right)$$

$$\underline{I=\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x-\cos x+C}$$

50. $I=\int\frac{dx}{x\sqrt{x^2+x+2}}\Rightarrow$

$$\left\{\begin{matrix}x^2+x+2=(u-x)^2=u^2-2ux-x^2\Rightarrow x=\frac{u^2-2}{1+2u}\Rightarrow \underline{u=\sqrt{x^2+x+2}+x}\\ dx=\frac{(1+2u)2u-2\left(u^2-2\right)}{(1+2u)^2}du=\frac{2\left(u^2+u+2\right)}{(1+2u)^2}du\\ \sqrt{x^2+x+2}=(u-x)=u-\frac{u^2-2}{1+2u}=\frac{u^2+u+2}{1+2u}\end{matrix}\right\}$$

$$I=\int\frac{dx}{x\sqrt{x^2+x+2}}=\int\frac{\frac{2\left(u^2+u+2\right)}{(1+2u)^2}du}{\frac{u^2-2}{1+2u}\cdot\frac{u^2+u+2}{1+2u}}=2\int\frac{du}{u^2-2}\Rightarrow\left\{\begin{array}{c}\text{Tipski integral:}\\ \int\frac{dk}{k^2-a^2}=\frac{1}{2a}\ln\left|\frac{k-a}{k+a}\right|\end{array}\right\}$$

$$I=2\frac{1}{2\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|=\frac{1}{\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|=\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{x^2+x+2}+x-\sqrt{2}}{\sqrt{x^2+x+2}+x+\sqrt{2}}\right|$$

51. $$I=\int x^3e^{2x}dx\Rightarrow\left\{\begin{array}{cc}u=x^3 & dv=e^{2x}\\ du=3x^2dx; & v=\int e^{2x}dx=\frac{1}{2}e^{2x}\end{array}\right\}\Rightarrow\int x^3e^{2x}dx=x^3\frac{1}{2}e^{2x}-\int\frac{1}{2}e^{2x}3x^2dx$$

$$I=x^3\frac{1}{2}e^{2x}-\frac{3}{2}\int x^2e^{2x}dx$$

$$\frac{3}{2}\int x^2e^{2x}dx\Rightarrow\left\{\begin{array}{cc}u=x^2 & dv=e^{2x}\\ du=2xdx; & v=\int e^{2x}dx=\frac{1}{2}e^{2x}\end{array}\right\}\Rightarrow\int x^2e^{2x}dx=x^2\frac{1}{2}e^{2x}-\int\frac{1}{2}e^{2x}2xdx$$

$$I=x^3\frac{1}{2}e^{2x}-\frac{3}{2}\left(x^2\frac{1}{2}e^{2x}-\int\frac{1}{2}e^{2x}2xdx\right)=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{2}\left(\int\frac{1}{2}e^{2x}2xdx\right)$$

$$\underline{I=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{2}\int xe^{2x}dx}$$

$$I=\int xe^{2x}dx\Rightarrow$$

$$\Rightarrow\left\{\begin{array}{cc}u=x & dv=e^{2x}\\ du=dx & v=\int e^{2x}dx=\frac{1}{2}e^{2x}\end{array}\right\}\Rightarrow\int xe^{2x}dx=x\frac{1}{2}e^{2x}-\int\frac{1}{2}e^{2x}dx=x\frac{1}{2}e^{2x}-\frac{1}{2}\int e^{2x}dx$$

$\underline{I=x\frac{1}{2}e^{2x}-\frac{1}{4}e^{2x}}$ Uvrstimo u trazeni integral:

$$I=I=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{2}\int xe^{2x}dx=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{2}\left(x\frac{1}{2}e^{2x}-\frac{1}{4}e^{2x}\right)$$

$$I=\underline{\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{4}xe^{2x}-\frac{3}{8}e^{2x}+C}$$

52. $$I=\int\cos^3\left(\frac{x}{3}\right)dx=\int\left(1-\sin^2\left(\frac{x}{3}\right)\right)\cos\left(\frac{x}{3}\right)dx=\int\cos\frac{x}{3}dx-\int\sin^2\left(\frac{x}{3}\right)\cos\frac{x}{3}dx$$

$I=3\sin\frac{x}{3}-3\int\sin^2\left(\frac{x}{3}\right)\frac{1}{3}\cos\frac{x}{3}dx$ Koristimo steceno znanje:

$$I=3\sin\frac{x}{3}-3\frac{1}{3}\sin^3\frac{x}{3}=\underline{3\sin\frac{x}{3}-\sin^3\frac{x}{3}+C}$$

53. $$I=\int\frac{x}{(x+2)(x+3)}dx=\left\{\begin{array}{c}\frac{x}{(x+2)(x+3)}=\frac{A}{(x+2)}+\frac{B}{(x+3)}\\ \underline{x=A(x+3)+B(x+2)}\end{array}\right\}=$$

$$=\left\{\begin{array}{l}x=-2\rightarrow -2=A(1)+B(0)\rightarrow A=-2\\ x=-3\rightarrow -3=A(0)+B(-1)\rightarrow B=3\end{array}\right\}$$

$$I=\int\frac{x}{(x+2)(x+3)}dx=\int\left(\frac{-2}{(x+2)}+\frac{3}{(x+3)}\right)dx=-2\left(\ln|x+2|+3\ln|x+3|\right)$$

$$\underline{I=\ln\frac{\left|(x+3)^3\right|}{\left|(x+2)^2\right|}+C}$$

54. $$I=\int\frac{x}{\sqrt{\left(5-4x-x^2\right)^3}}dx\Rightarrow$$

$$\Rightarrow\left\{\begin{array}{l}5-4x-x^2=(5+x)(1-x)=(1-x)^2u^2\Rightarrow(5+x)=(1-x)u^2\Rightarrow x=\frac{u^2-5}{1+u^2}\\ dx=\frac{2u\left(1+u^2\right)-\left(u^2-5\right)2u}{\left(1+u^2\right)^2}du=\frac{2u+2u^3-2u^3+10u}{(1+u)^2}du=\frac{12u}{(1+u)^2}du\\ \sqrt{5-4x-x^2}=(1-x)u=u-\frac{u^2-5}{1+u^2}u=\frac{6u}{1+u^2}\end{array}\right\}$$

$$I=\int\frac{x}{\sqrt{\left(5-4x-x^2\right)^3}}dx=\int\frac{\frac{u^2-5}{\left(1+u^2\right)}\left(\frac{12u}{(1+u)^2}\right)du}{\left(\frac{6u}{1+u^2}\right)^3}=12\int\frac{u\left(u^2-5\right)}{216u^3}du$$

$$I=\frac{12}{216}\int\left(1-\frac{5}{u^2}\right)du=\frac{1}{18}\left(\int du-\int 5u^{-2}du\right)=\frac{1}{18}\left(u+\frac{5}{u}\right)=$$

$$I=\frac{1}{18}\left(\frac{\sqrt{5-4x-x^2}}{1-x}+\frac{5(1-x)}{\sqrt{5-4x-x^2}}\right)=\frac{1}{18}\left[\frac{5-4x-x^2+5(1-x)^2}{(1-x)\sqrt{5-4x-x^2}}\right]$$

$$I=\frac{1}{18}\left[\frac{5-4x-x^2+5\left(1-2x+x^2\right)}{(1-x)\sqrt{5-4x-x^2}}\right]=\frac{1}{18}\left[\frac{5-4x-x^2+5-10x+5x^2}{(1-x)\sqrt{5-4x-x^2}}\right]$$

$$I=\frac{1}{18}\left[\frac{4x^2-14x+10}{(1-x)\sqrt{5-4x-x^2}}\right]=\frac{1}{18}\left[\frac{(x-1)\left(x-\frac{5}{2}\right)}{(1-x)\sqrt{5-4x-x^2}}\right]=\frac{1}{18}\left[\frac{-\left(x-\frac{5}{2}\right)}{\sqrt{5-4x-x^2}}\right]$$

$$I=\frac{1}{18}\left[\frac{\frac{5-2x}{2}}{\sqrt{5-4x-x^2}}\right]=\underline{\frac{5-2x}{9\sqrt{5-4x-x^2}}+C}$$

55. $$I=\int\frac{dx}{\sqrt{1-\sin 2x}}\Rightarrow\left\{\begin{array}{c}\sin^2 x=\frac{1-\cos 2x}{2}\\ \sin^2\left(\frac{\pi}{4}-x\right)=\frac{1-\cos\left(\frac{\pi}{2}-2x\right)}{2}\end{array}\right\}=\int\frac{dx}{\sqrt{1-\cos\left(\frac{\pi}{2}-2x\right)}}$$

$$I=\frac{\sqrt{2}}{2}\int\frac{dx}{\sin\left(\frac{\pi}{4}-x\right)}=\int\frac{dx}{\sqrt{1-\sin 2x}}=dx=\int\csc\left(\frac{\pi}{4}-x\right)dx$$

$$I=\underline{-\frac{\sqrt{2}}{2}\ln\left|\csc\left(\frac{\pi}{4}-x\right)-\cot\left(\frac{\pi}{4}-x\right)\right|+C}$$

56. $$I=\int\frac{dx}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\Rightarrow\left\{\begin{array}{c}u=x^{\frac{1}{6}}\quad x=u^6\to x^{\frac{1}{2}}=u^3;x^{\frac{1}{3}}=u^2\\ dx=6u^5du\end{array}\right\}\Rightarrow\int\frac{6u^5}{u^3+u^2}du=6\int\frac{u^3}{u+1}du$$

$$\left\{u^3:(u+1)=u^2-u+1-\frac{1}{u+1}\right\}\Rightarrow 6\int\frac{u^3}{u+1}du=6\int\left(u^2-u+1-\frac{1}{u+1}\right)dx$$

$$I=6\left(\frac{u^3}{3}-\frac{u^2}{2}+u-\ln|u+1|\right)+C=\underline{2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\ln\left|x^{\frac{1}{6}}-1\right|+C}$$

57. $$I=\int\frac{x^2+2}{x(x+2)(x-1)}dx\Rightarrow\left\{\begin{array}{c}\frac{x^2+2}{x(x+2)(x-1)}=\frac{A}{x}+\frac{B}{(x+2)}+\frac{C}{(x-1)}\\ \underline{x^2+2=A(x+2)(x-1)+Bx(x-1)+Cx(x+2)}\end{array}\right\}$$

$$=\left\{\begin{array}{c}x=0\to 2=A(2)(-1)+B(0)(-1)+C(0)(2)\to A=-1\\ x=-2\to 6=A(0)(-3)+B(-2)(-3)+C(0)(0)\to B=1\\ x=1\to 3=A(3)(0)+B(1)(0)+C(1)(3)\to C=1\end{array}\right\}\Rightarrow$$

$$I=\int\frac{x^2+2}{x(x+2)(x-1)}dx=\int\left(\frac{1}{x}+\frac{1}{(x+2)}+\frac{1}{(x-1)}\right)dx=\underline{\ln|x|+\ln|x+2|+\ln|x-1|+C}$$

58. $I=\int\frac{\sqrt{9-4x^2}}{x}dx \Rightarrow \left\{\begin{array}{c} \sqrt{9-4x^2}=\sqrt{4\left(\frac{9}{4}-x^2\right)}=2\sqrt{\left(\frac{9}{4}-x^2\right)} \\ x=\frac{3}{2}\sin\varphi \rightarrow\ dx=\frac{3}{2}\cos\varphi\, d\varphi \\ \sqrt{9-4x^2}=\sqrt{9-9\sin^2\varphi}=3\sqrt{1-\sin^2\varphi}=3\sqrt{\cos^2\varphi}=3\cos\varphi \end{array}\right\}$

$$I=\int\frac{\sqrt{9-4x^2}}{x}dx=\int\frac{3\cos\varphi}{\frac{3}{2}\sin\varphi}\cdot\frac{3}{2}\cos\varphi\, d\varphi=3\int\frac{\cos^2\varphi}{\sin\varphi}d\varphi=3\int\frac{1-\sin^2\varphi}{\sin\varphi}d\varphi=$$

$$I=3\int\left(\frac{1}{\sin\varphi}-\sin\varphi\right)d\varphi=3\int\left(\csc\varphi-\sin\varphi\right)d\varphi=3\left(\ln\left|\csc\varphi-\cot\varphi\right|+\cos\varphi\right)+C_1$$

$$\Rightarrow \text{Zamijenimo}:\left\{\begin{array}{c} \csc\varphi=\frac{1}{\sin\varphi}=\frac{3}{2x} \\ \cot\varphi=\frac{\cos\varphi}{\sin\varphi}=\frac{\frac{\sqrt{9-4x^2}}{3}}{\frac{2x}{3}}=\frac{\sqrt{9-4x^2}}{2x} \end{array}\right\}$$

$$I=3\ln\left|\frac{3}{2x}-\frac{\sqrt{9-4x^2}}{2x}\right|+3\frac{\sqrt{9-4x^2}}{3}+C_2=\underline{3\ln\left|\frac{3-\sqrt{9-4x^2}}{2x}\right|+\sqrt{9-4x^2}+C}$$