Function (mathematics)

Dodatak 1

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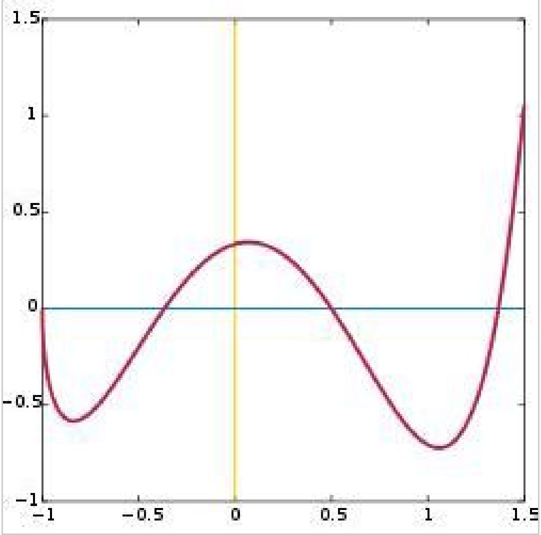
The mathematical concept of a **function** expresses the intuitive idea that one quantity (the argument of the function, also known as the input) completely determines another quantity (the value, or the output). A function assigns a unique value to each input of a specified type. The argument and the value may be real numbers, but they can also be elements from any given sets: the domain and the codomain of the function. An example of a function with the real numbers as both its domain and codomain is the function $f(x) = 2x$, which assigns to every real number the real number with twice its value. In this case, it is written that $f(5) = 10$.



Graph of example function,
$$f(x) = \frac{(4x^3-6x^2+1)\sqrt{x+1}}{3-x}$$
Both the domain and the range in the picture are the set of real numbers between -1 and 1.5.

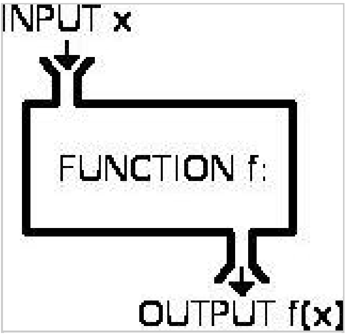
In addition to elementary functions on numbers, functions include maps between algebraic structures like groups and maps between geometric objects like manifolds. In the abstract set-theoretic approach, a function is a relation between the domain and the codomain that associates each element in the domain with exactly one element in the codomain. An example of a function with domain {A,B,C} and codomain {1,2,3} associates A with 1, B with 2, and C with 3.

There are many ways to describe or represent functions: by a formula, by an algorithm that computes it, by a plot or a graph. A table of values is a common way to specify a function in statistics, physics, chemistry, and other sciences. A function may also be described through its relationship to other functions, for example, as the inverse function or a solution of a differential equation. There are uncountably many different functions from the set of natural numbers to itself, most of which cannot be expressed with a formula or an algorithm.

In a setting where they have numerical outputs, functions may be added and multiplied, yielding new functions. Collections of functions with certain properties, such as continuous functions and differentiable functions, usually required to be closed under certain operations, are called function spaces and are studied as objects in their own right, in such disciplines as real analysis and complex analysis. An important operation on functions, which distinguishes them from numbers, is the composition of functions.

Overview

Because functions are so widely used, many traditions have grown up around their use. The symbol for the input to a function is often called the **independent variable** or **argument** and is often represented by the letter x or, if the input is a particular time, by the letter t. The symbol for the output is called the **dependent variable** or **value** and is often represented by the letter y. The function itself is most often called f, and thus the notation $y = f(x)$ indicates that a function named f has an input named x and an output named y.



A function f takes an input, x, and returns an output $f(x)$. One metaphor describes the function as a "machine" or "black box" that converts the input into the output.

The set of all permitted inputs to a given function is called the domain of the function. The set of all resulting outputs is called the image or range of the function. The image is often a subset of some larger set, called the codomain of a function. Thus, for example, the function $f(x) = x^2$ could take as its domain the set of all real numbers, as its image the set of all non-negative real numbers, and as its codomain the set of all real numbers. In that case, we would describe f as a real-valued function of a real variable. Sometimes, especially in computer science, the term "range" refers to the codomain rather than the image, so care needs to be taken when using the word.

It is usual practice in mathematics to introduce functions with temporary names like f. For example, $f(x) = 2x+1$, implies $f(3) = 7$; when a name for the function is not needed, the form $y = 2x + 1$ may be used. If a function is often used, it may be given a more permanent name as, for example,

$$\mathrm{Square}(x) = x^2.$$

Functions need not act on numbers: the domain and codomain of a function may be arbitrary sets. One example of a function that acts on non-numeric inputs takes English words as inputs and returns the first letter of the input word as output. Furthermore, functions need not be described by any expression, rule or algorithm: indeed, in some cases it may be impossible to define such a rule. For example, the association between inputs and outputs in a choice function often lacks any fixed rule, although each input element is still associated to one and only one output.

A function of two or more variables is considered in formal mathematics as having a domain consisting of ordered pairs or tuples of the argument values. For example Sum(x,y) = x+y operating on integers is the function Sum with a domain consisting of pairs of integers. Sum then has a domain consisting of elements like (3,4), a codomain of integers, and an association between the two that can be described by a set of ordered pairs like ((3,4), 7). Evaluating Sum(3,4) then gives the value 7 associated with the pair (3,4).

A family of objects indexed by a set is equivalent to a function. For example, the sequence *1, 1/2, 1/3, ..., 1/n,* ... can be written as the ordered sequence <*1/n*> where n is a natural number, or as a function f(n) = 1/n from the set of natural numbers into the set of rational numbers.

Dually, a surjective function partitions its domain into disjoint sets indexed by the codomain. This partition is known as the kernel of the function, and the parts are called the fibers or level sets of the function at each element of the codomain. (A non-surjective function divides its domain into disjoint and possibly-empty subsets).

Definition

One precise definition of a function is that it consists of an ordered triple of sets, which may be written as (X, Y, F). X is the domain of the function, Y is the codomain, and F is a set of ordered pairs. In each of these ordered pairs (a, b), the first element a is from the domain, the second element b is from the codomain, and every element in the domain is the first element in one and only one ordered pair. The set of all b is known as the image of the function. Some authors use the term "range" to mean the image, others to mean the codomain.

The notation f:X→Y indicates that f is a function with domain X and codomain Y.

In most practical situations, the domain and codomain are understood from context, and only the relationship between the input and output is given. Thus

$$(\mathbb{R}, \mathbb{R}, \{(x, x^2) : x \in \mathbb{R}\})$$

is usually written as

$$y = x^2.$$

The graph of a function is its set of ordered pairs. Such a set can be plotted on a pair of coordinate axes; for example, (3, 9) is the point of intersection of the lines x = 3 and y = 9.

A function is a special case of a more general mathematical concept, the relation, for which the restriction that each element of the domain appear as the first element in one and only one ordered pair is removed (or, in other words, the restriction that each input be associated to exactly one output). A relation is "single-valued" or "functional" when for each element of the domain set, the graph contains at most one ordered pair (and possibly none) with it as a first element. A relation is called "left-total" or simply "total" when for each element of the domain, the graph contains at least one ordered pair with it as a first element (and possibly more than one). A relation that is both left-total and single-valued is a function.

In some parts of mathematics, including recursion theory and functional analysis, it is convenient to study partial functions in which some values of the domain have no association in the graph; i.e., single-valued relations. For example, the function f such that f(x) = 1/x does not define a value for x = 0, and so is only a partial function from the real line to the real line. The term *total function* can be used to stress the fact that every element of the domain does appear as the first element of an ordered pair in the graph. In other parts of mathematics, non-single-valued relations are similarly conflated with functions: these are called multivalued functions, with the corresponding term single-valued function for ordinary functions.

Some authors (especially in set theory) define a function as simply its graph f, with the restriction that the graph should not contain two distinct ordered pairs with the same first element. Indeed, given such a graph, one can construct a suitable triple by taking the set of all first elements as the domain and the set of all second elements as the codomain: this automatically causes the function to be total and surjective . However, most authors in advanced mathematics outside of set theory prefer the greater power of expression[*citation needed*] afforded by defining a function as an ordered triple of sets.

Many operations in set theory—such as the power set—have the class of all sets as their domain, therefore, although they are informally described as functions, they do not fit the set-theoretical definition above outlined.

Vocabulary

A specific input in a function is called an **argument** of the function. For each argument value x, the corresponding unique y in the codomain is called the function **value** at x, **output** of f for an argument x, or the **image** of x **under** f. The image of x may be written as $f(x)$ or as y.

The **graph** of a function f is the set of all ordered pairs $(x, f(x))$, for all x in the domain X. If X and Y are subsets of **R**, the real numbers, then this definition coincides with the familiar sense of "graph" as a picture or plot of the function, with the ordered pairs being the Cartesian coordinates of points.

A function can also be called a **map** or a **mapping**. Some authors, however, use the terms "function" and "map" to refer to different types of functions. Other specific types of functions include **functionals** and **operators**.

Notation

Formal description of a function typically involves the function's name, its domain, its codomain, and a rule of correspondence. Thus we frequently see a two-part notation, an example being

$$\begin{aligned} f\colon \mathbb{N} &\to \mathbb{R} \\ n &\mapsto \frac{n}{\pi} \end{aligned}$$

where the first part is read:

- "f is a function from **N** to **R**" (one often writes informally "Let $f: X \to Y$" to mean "Let f be a function from X to Y"), or
- "f is a function on **N** into **R**", or
- "f is an **R**-valued function of an **N**-valued variable",

and the second part is read:

- n maps to $\frac{n}{\pi}$.

Here the function named "f" has the natural numbers as domain, the real numbers as codomain, and maps n to itself divided by π. Less formally, this long form might be abbreviated

$$f(n) = \frac{n}{\pi},$$

where $f(n)$ is read as "f as function of n" or "f of n". There is some loss of information: we no longer are explicitly given the domain **N** and codomain **R**.

It is common to omit the parentheses around the argument when there is little chance of confusion, thus: sin x; this is known as prefix notation. Writing the function after its argument, as in $x\ f$, is known as postfix notation; for example, the factorial function is customarily written $n!$, even though its generalization, the gamma function, is written $\Gamma(n)$. Parentheses are still used to resolve ambiguities and denote precedence, though in some formal settings the consistent use of either prefix or postfix notation eliminates the need for any parentheses.

Functions with multiple inputs and outputs

The concept of function can be extended to an object that takes a combination of two (or more) argument values to a single result. This intuitive concept is formalized by a function whose domain is the Cartesian product of two or more sets.

For example, consider the function that associates two integers to their product: $f(x, y) = x \cdot y$. This function can be defined formally as having domain **Z**×**Z**, the set of all integer pairs; codomain **Z**; and, for graph, the set of all pairs $((x,y), x \cdot y)$. Note that the first component of any such pair is itself a pair (of integers), while the second component is a single integer.

The function value of the pair (x,y) is $f((x,y))$. However, it is customary to drop one set of parentheses and consider $f(x,y)$ a **function of two variables**, x and y. Functions of two variables may be plotted on the three-dimensional Cartesian as ordered triples of the form $(x,y,f(x,y))$.

The concept can still further be extended by considering a function that also produces output that is expressed as several variables. For example, consider the function swap$(x, y) = (y, x)$ with domain **R**×**R** and codomain **R**×**R** as well. The pair (y, x) is a single value in the codomain seen as a Cartesian product.

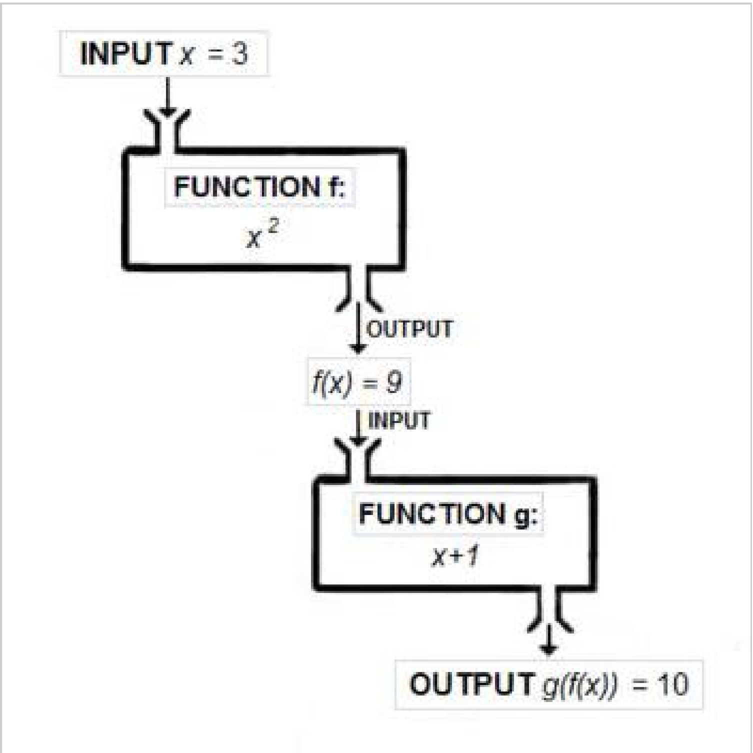
Injective and surjective functions

Three important kinds of function are the **injections** (or **one-to-one functions**), which have the property that if $f(a) = f(b)$ then a must equal b; the **surjections** (or **onto functions**), which have the property that for every y in the codomain there is an x in the domain such that $f(x) = y$; and the **bijections**, which are both one-to-one and onto. This nomenclature was introduced by the Bourbaki group.

When the definition of a function by its graph only is used, since the codomain is not defined, the "surjection" must be accompanied with a statement about the set the function maps onto. For example, we might say f maps onto the set of all real numbers.

Function composition

Main article: Function composition



A composite function $g(f(x))$ can be visualized as the combination of two "machines". The first takes input x and outputs $f(x)$. The second takes $f(x)$ and outputs $g(f(x))$.

The **function composition** of two or more functions takes the output of one or more functions as the input of others. The functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ can be *composed* by first applying f to an argument x to obtain $y = f(x)$ and then applying g to y to obtain $z = g(y)$. The composite function formed in this way from general f and g may be written

$$\begin{aligned} g \circ f \colon X &\rightarrow Z \\ x &\mapsto g(f(x)). \end{aligned}$$

This notation follows the form such that

$$(g \circ f)(x) = g(f(x)).$$

The function on the right acts first and the function on the left acts second, reversing English reading order. We remember the order by reading the notation as "g of f". The order is important, because rarely do we get the same result both ways. For example, suppose $f(x) = x^2$ and $g(x) = x+1$. Then $g(f(x)) = x^2+1$, while $f(g(x)) = (x+1)^2$, which is x^2+2x+1, a different function.

In a similar way, the function given above by the formula $y = 5x-20x^3+16x^5$ can be obtained by composing several functions, namely the addition, negation, and multiplication of real numbers.

An alternative to the colon notation, convenient when functions are being composed, writes the function name above the arrow. For example, if f is followed by g, where g produces the complex number e^{ix}, we may write

$$\mathbb{N}\xrightarrow{f}\mathbb{R}\xrightarrow{g}\mathbb{C}.$$

A more elaborate form of this is the commutative diagram.

Identity function

Main article: Identity function

The unique function over a set X that maps each element to itself is called the **identity function** for X, and typically denoted by id_X. Each set has its own identity function, so the subscript cannot be omitted unless the set can be inferred from context. Under composition, an identity function is "neutral": if f is any function from X to Y, then

$$f \circ \mathrm{id}_X = f,$$
$$\mathrm{id}_Y \circ f = f.$$

Restrictions and extensions

Informally, a **restriction** of a function f is the result of trimming its domain.

More precisely, if f is a function from a X to Y, and S is any subset of X, the **restriction of f to S** is the function $f|_S$ from S to Y such that $f|_S(s) = f(s)$ for all s in S.

If g is a restriction of f, then it is said that f is an **extension** of g.

The **overriding** of $f: X \rightarrow Y$ by $g: W \rightarrow Y$ (also called **overriding union**) is an extension of g denoted as $(f \oplus g): (X \cup W) \rightarrow Y$. Its graph is the set-theoretical union of the graphs of g and $f|_{X \setminus W}$. Thus, it relates any element of the domain of g to its image under g, and any other element of the domain of f to its image under f. Overriding is an associative operation; it has the empty function as an identity element. If $f|_{X \cap W}$ and $g|_{X \cap W}$ are pointwise equal (e.g., the domains of f and g are disjoint), then the **union** of f and g is defined and is equal to their overriding union. This definition agrees with the definition of union for binary relations.

Inverse function

Main article: Inverse function

If f is a function from X to Y then an **inverse function** for f, denoted by f^{-1}, is a function in the opposite direction, from Y to X, with the property that a round trip (a composition) returns each element to itself. Not every function has an inverse; those that do are called **invertible**. The inverse function exists if and only if f is a bijection.

As a simple example, if f converts a temperature in degrees Celsius C to degrees Fahrenheit F, the function converting degrees Fahrenheit to degrees Celsius would be a suitable f^{-1}.

$$f(C) = \frac{9}{5}C + 32$$
$$f^{-1}(F) = \frac{5}{9}(F - 32)$$

The notation for composition is similar to multiplication; in fact, sometimes it is denoted using juxtaposition, gf, without an intervening circle. With this analogy, identity functions are like the multiplicative identity, 1, and inverse functions are like reciprocals (hence the notation).

For functions that are injections or surjections, generalized inverse functions can be defined, called left and right inverses respectively. Left inverses map to the identity when composed to the left; right inverses when composed to the right.