Granična vrijednost u beskonačnosti

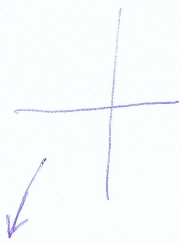
Limese u $+\infty$ riješavamo istim »trikovima« kao na seminaru 8 i 9. Vidi dodatak 5, riješene primjere 230, 245 i 257.

Limes u $-\infty$ pokušavamo svesti na limes u $+\infty$ zamjenom varijabli $y = -x$ ili jednostavnom zamjenom $x \mapsto (-x)$. Vidi riješene primjere 233, 234 i 247 dodatka 5.

Primjer 1. $\lim\limits_{x\to-\infty} \dfrac{2x+3}{x+\sqrt{x^2-x}} = \left\{ \begin{matrix} y=-x \\ x=-y \end{matrix} \right\} = \lim\limits_{y\to\infty} \dfrac{-2y+3}{-y+\sqrt{y^2+y}} = \left[\dfrac{-\infty}{-\infty+\infty}\right] = \lim\limits_{y\to\infty} \dfrac{-2+\frac{3}{y}}{-1+\sqrt{1+\frac{1}{y}}} \cdot \dfrac{1+\sqrt{1+\frac{1}{y}}}{1+\sqrt{1+\frac{1}{y}}} = \left[\dfrac{-2}{0}\right] =$

$$= \lim_{y\to\infty} \frac{\left(-2+\frac{3}{y}\right)\left(1+\sqrt{1+\frac{1}{y}}\right)}{-1+1+\frac{1}{y}} = \lim_{y\to\infty} y\left(-2+\frac{3}{y}\right)\left(1+\sqrt{1+\frac{1}{y}}\right) = +\infty\cdot(-2)\cdot 1 = -\infty$$

$$= \lim_{y\to\infty} \frac{(-2y+3)(y+\sqrt{y^2+y})}{-y^2+y^2+y} = \lim_{y\to\infty}\left(-2+\frac{3}{y}\right)\left(y+\sqrt{y^2+y}\right) = -2(+\infty+\infty) = -2(+\infty) = -\infty$$

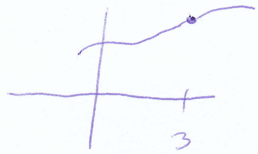


Granična vrijednost u točki

Jednostavni slučaj: ako se limes promatra u točki gdje je izrazom zadana funkcija neprekidna, dovoljno je uvrstiti točku u izraz.

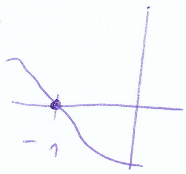
Primjer 2. $\lim\limits_{x\to3} \dfrac{x^3-3}{x^2-2x+3} = \dfrac{3^3-3}{3^2-2\cdot3+3} = \dfrac{24}{9-6+3} = \dfrac{24}{6} = 4$

Zapazi da je $\dfrac{x^3-3}{x^2-2x+3}$ *neprekidno u* $x = 3$



Primjer 3. $\lim\limits_{x\to-1} \dfrac{x^3+1}{2x^2-x-5} = \dfrac{-1+1}{2+1-5} = \dfrac{0}{-2} = 0$

Zapazi da je $\dfrac{x^3+1}{2x^2-x-5}$ *neprekidno u* $x = -1$

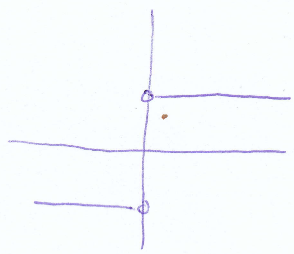


Složeniji slučajevi: kada funkcija dana izrazom nije neprekidna u nuli odvojeno promatrati $\lim\limits_{x\to a+}$ i $\lim\limits_{x\to a-}$.

Primjer 4. $\lim\limits_{x\to0} \dfrac{|x|}{x} = N/P$ $\dfrac{|x|}{x}$ ima ~~pa~~ nije neprekidna u nuli

$$\lim_{x\to0_+} \frac{|x|}{x} = \lim_{x\to0_+} \frac{x}{x} = \lim_{x\to0_+} 1 = 1$$

$$\lim_{x\to0_-} \frac{|x|}{x} = \lim_{x\to0_+} \frac{-x}{x} = \lim_{x\to0_-} (-1) = -1$$

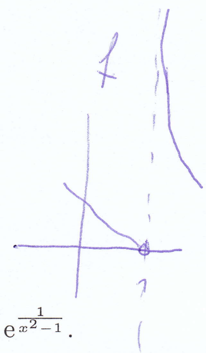


Primjer 5. Ispitati $\lim_{x\to 1} e^{\frac{1}{x^2-1}}$ $= N/P$ *Pažljivo! Kada je $x = 1$ tada $x^2 - 1 = 0$.*

$e^{\frac{1}{x^2-1}}$ nije neprekidna u nuli i nije def za $x=1$

$$\lim_{x\to 1+} e^{\frac{1}{x^2-1}} = \lim_{x\to 1+} e^{\frac{1}{(x-1)(x+1)}} = e^{\lim_{x\to 1+}\frac{1}{(x-1)(x+1)}} = e^{\frac{1}{(0+)(2)}} = e^{+\infty} = +\infty$$

$$\lim_{x\to 1-} e^{\frac{1}{x^2-1}} = e^{\lim_{x\to 1-}\frac{1}{(x-1)(x+1)}} = e^{\frac{1}{(0-)(2)}} = e^{\frac{1}{0-}} = e^{-\infty} = 0$$



Dio izraza $\frac{1}{x^2-1}$ nije neprekidna funkcija za $x = 1$ i stoga smo bili prisiljeni odvojeno ispitivati $\lim_{x\to 1+} e^{\frac{1}{x^2-1}}$ i $\lim_{x\to 1-} e^{\frac{1}{x^2-1}}$.

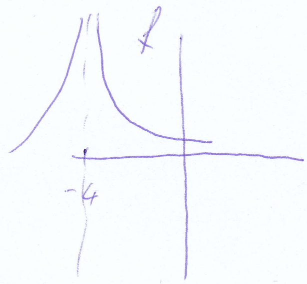
U sljedećem primjeru otkriveni singularitet a lakše je promatrati ako ga preselimo u nulu zamjenom $x \mapsto t - a$.

Primjer 6. $\lim_{x\to -4} \frac{x^2-10}{x^2+8x+16} = \left[\frac{6}{0}\right] =?$ $\left\{\begin{matrix} x \longmapsto t-4 \\ x \to -4 \\ t \to 0 \end{matrix}\right\}$ *Pažljivo! Razdvoji na limes zdesna i slijeva.*

$$= \lim_{t\to 0} \frac{(t-4)^2-10}{(t-4)^2+8(t-4)+16} = \lim_{t\to 0} \frac{t^2-8t+6}{t^2} = +\infty$$

$$\lim_{t\to 0+} \frac{t^2-8t+6}{t^2} = \frac{6}{0+} = +\infty$$

$$\lim_{t\to 0-} \frac{t^2-8t+6}{t^2} = \frac{6}{(0-)^2} = \frac{6}{0+} = +\infty$$

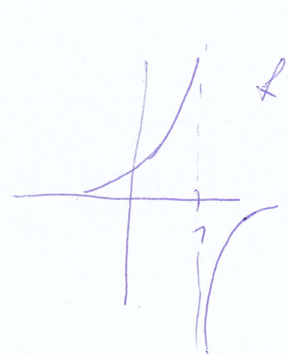


Dovoljno vješti studenti mogu slične primjere računati i bez zamjene varijabli.

Primjer 7. $\lim_{x\to 1} \frac{x^2}{x^2-1} =$ $\left[\frac{1}{0}\right] =$ VIDI ISPOD $= N/P$

$$\lim_{x\to 1+} \frac{x^2}{x^2-1} = \lim_{x\to 1+} \frac{x^2}{(x-1)(x+1)} = \frac{1}{(0+)\cdot 2} = \frac{1}{0+} = +\infty$$

$$\lim_{x\to 1-} \frac{x^2}{x^2-1} = \lim_{x\to 1-} \frac{x^2}{(x-1)(x+1)} = \frac{1}{(0-)\cdot 2} = \frac{1}{0-} = -\infty$$



Limese $x \to a$ slijeva i zdesna možemo provjeriti tako da u izraz uvrstimo vrijednosti vrlo blizu a (ispod i iznad).

Limes racionalne funkcije oblika $\lim\limits_{x\to a}\dfrac{p(x)}{q(x)}=\dfrac{0}{0}$**:** pojednostavljuje se faktoriziranjem ili dijeljenjem brojnika i nazivnika sa $(x-a)$.

Primjer 8. $\lim\limits_{x\to-2}\dfrac{x^2-4}{x^3+2x^2+x+2} = \lim\limits_{x\to-2}\dfrac{x-2}{x^2+1} = \dfrac{-4}{5}$

$f(x)$

$(x^2-4):(x+2) = x-2$

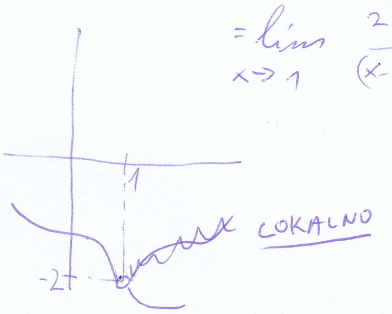
$(x^3+2x^2+x+2):(x+2) = x^2+1$
$-x^3+2x^2$
$0 \quad x+2$

Limes iracionalne funkcije oblika $\dfrac{0}{0}$**:** ponekad se može svesti na prethodni oblik racionalizacijom ili pogodnom supstitucijom. Općenito je ove limese mnogo lakše riješiti korištenjem L'Hopitalovog pravila (vidi seminar 12).

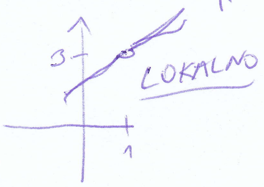
Primjer 9. $\lim\limits_{x\to1}\dfrac{\sqrt{2-x^2}-x}{x-1} = \left[\dfrac{\sqrt{1}-1}{0}=\dfrac{0}{0}\right] = \lim\limits_{x\to1}\dfrac{\sqrt{2-x^2}-x}{x-1}\cdot\dfrac{\sqrt{2-x^2}+x}{\sqrt{2-x^2}+x} =$ *Racionalizacijom!*

$= \lim\limits_{x\to1}\dfrac{2-2x^2}{(x-1)(\sqrt{2-x^2}+x)} = \left[\dfrac{0}{0\cdot2}\right] = \lim\limits_{x\to1}\dfrac{\cancel{(x-1)}(-2x-2)}{\cancel{(x-1)}(\sqrt{2-x^2}+x)} = \dfrac{-4}{2} = -2$

$(-2x^2+2):(x-1) = -2x-2$
$\mp 2x^2 \pm 2x$
$-2x+2$
$=$



Primjer 10. $\lim\limits_{x\to1}\dfrac{x-1}{\sqrt[3]{x}-1} = \left\{\dfrac{1-1}{1-1}=\dfrac{0}{0}\right\} = \left\{\begin{array}{l}x=t^3\\ t=\sqrt[3]{x}\\ x\to1\\ t\to\sqrt[3]{1}=1\end{array}\right\} = \lim\limits_{t\to1}\dfrac{t^3-1}{t-1} = \lim\limits_{t\to1}\dfrac{\cancel{(t-1)}(t^2+t+1)}{\cancel{t-1}} = 3$ *Zamjena $x=t^3$!*



$(t^3-1):(t-1) = t^2+t+1$
$-t^3 \mp t^2$
t^2-1
$-t^2 \mp t$
$t-1$

Nekoliko zadataka za samostalnu vježbu s rješenjem.

Dodatak 5: 238-242, 248-252.

$\lim\limits_{x\to\pm\infty}\dfrac{\sqrt{x^2+1}}{x+1}=\pm1$

$\lim\limits_{x\to\infty}\dfrac{\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}}{\sqrt{2x+1}}=\dfrac{1}{\sqrt{2}}$

$\lim\limits_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)=\dfrac{5}{2}$

$\lim\limits_{x\to1}\dfrac{x^2-1}{2x^2-x-1}=\dfrac{2}{3}$

$\lim\limits_{x\to2}\dfrac{x^3-3x+2}{x^4-4x+3}=\dfrac{4}{11}$

$\lim\limits_{x\to1}\dfrac{x^3+x^2+x-3}{x^3-3x^2+4x-2}=6$

$\lim\limits_{x\to2}\dfrac{x^3-3}{x^3-2x-4}=\text{N/P}$

$\lim\limits_{x\to2}\dfrac{x^3-2x+3}{x^2-4x+4}=+\infty$

$\lim\limits_{x\to1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right)=-1$

$\lim\limits_{x\to0}\left(\dfrac{\sqrt{5+x}-\sqrt{5}}{x}\right)=\dfrac{1}{2\sqrt{5}}$

$\lim\limits_{x\to0}\left(\dfrac{\sqrt{1-2x-x^2}-(1+x)}{x}\right)=-2$

$\lim\limits_{x\to3}\left(\dfrac{\sqrt{6+x}-3}{x-3}\right)=\dfrac{1}{6}$

$\lim\limits_{x\to-1}\left(\dfrac{\sqrt{x^2-8x+7}-3+x}{x^2+4x+3}\right)=-\dfrac{1}{8}$

$\lim\limits_{x\to0}\left(\dfrac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}\right)=\dfrac{4}{3}$