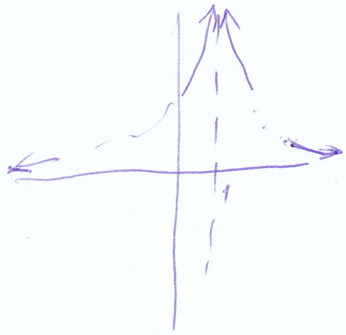
Glavni dio seminara

U sljedećim primjerima za dane funkcije treba odrediti:

- domenu,
- asimptote: *ispituju se na rubovima domene ili u točkama prekida*,
- globalna svojstva
  - omeđenost: *zaključak izvodimo iz asimptota*,
  - parnost i neparnost: *uspoređujemo $f(-x)$ sa $f(x)$ i $-f(x)$*,
  - periodičnost: *na osnovi periodičnosti trigonometrijskih funkcija*.

Primjer 1. $f(x) = \dfrac{2}{\sqrt[3]{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$



$\lim\limits_{x\to+\infty} f(x) = \frac{1}{+\infty} = 0$

$\lim\limits_{x\to-\infty} f(x) = \frac{1}{+\infty} = 0$ H.A. $y=0$

$\lim\limits_{x\to 1_+} f(x) = \frac{2}{0_+} = +\infty$

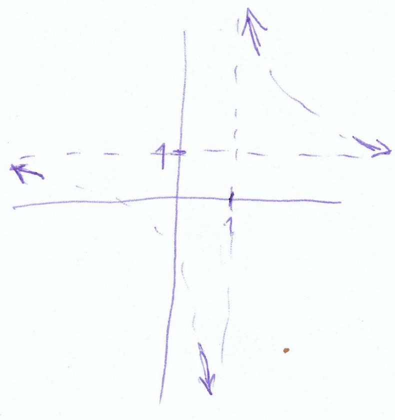
$\lim\limits_{x\to 1_-} f(x) = \frac{2}{0_+} = +\infty$ $\Rightarrow$ neomeđena odozgo, omeđena odozdo $\Rightarrow$ V.A. $x=1$

$f(-x) = \dfrac{2}{\sqrt[3]{(-x-1)^2}} = \dfrac{2}{\sqrt[3]{(x+1)^2}} \neq f(x)$ nije parna

$\neq -f(x)$ nije neparna

nije kompozicija trig. funkcije $\Rightarrow$ nije periodična

Primjer 2. $f(x) = \dfrac{x}{x-1}$ $D(f) = \mathbb{R} \setminus \{1\}$



$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{x}{x-1} = 1$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to+\infty} \frac{-x}{-x-1} = 1$ H.A. $y=1$

$\lim\limits_{x\to 1_+} f(x) = \frac{1}{0_+} = +\infty$

$\lim\limits_{x\to 1_-} f(x) = \frac{1}{0_-} = -\infty$ neomeđena odozdo, neomeđena odozgo $\Rightarrow$ V.A. $x=1$

$f(-x) = \dfrac{-x}{-x-1} = \dfrac{x}{x+1} \neq f(x)$ nije parna

$\neq -f(x)$ nije neparna

nije komp. trig. f-je $\Rightarrow$ nije periodična

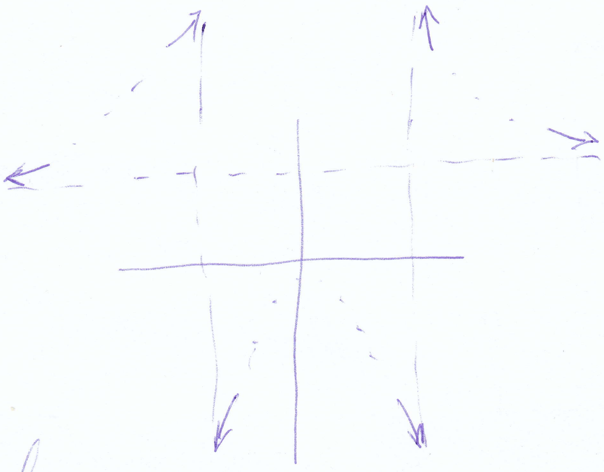
Primjer 3. $f(x) = \dfrac{x^2+1}{x^2-1}$

Vidi `http://lavica.fesb.hr/mat1/vjezbe/node107.html`

$x^2 - 1 \neq 0 \Rightarrow$

$(x-1)(x+1) \neq 0$

$\left.\begin{array}{l} x \neq +1 \\ x \neq -1 \end{array}\right\} \Rightarrow D(f) = \mathbb{R} \setminus \{-1, 1\}$



$\left.\begin{array}{l} \lim\limits_{x\to+\infty} \dfrac{x^2+1}{x^2-1} = 1 \\ \lim\limits_{x\to-\infty} f(x) = 1 \end{array}\right\}$ H.A. $y=1$

$\lim\limits_{x\to1_+} \dfrac{x^2+1}{(x-1)(x+1)} = \dfrac{2}{(0_+)\cdot 2} = \dfrac{2}{0_+} = +\infty$

$\lim\limits_{x\to1_-} \dfrac{x^2+1}{(x-1)(x+1)} = \dfrac{2}{(0_-)\cdot 2} = \dfrac{2}{0_-} = -\infty$

$\lim\limits_{x\to-1_+} \dfrac{x^2+1}{(x-1)(x+1)} = \dfrac{2}{(-2)(0_+)} = \dfrac{2}{0_-} = -\infty$

$\lim\limits_{x\to-1_-} \dfrac{x^2+1}{(x-1)(x+1)} = \dfrac{2}{(-2)(0_-)} = \dfrac{2}{0_+} = +\infty$

$\Rightarrow$ neomeđena odozdo, neomeđena odozgo, V.A. $x=1$, V.A. $x=-1$

$f(-x) = f(x)$ parna, nije neparna

nije trig.

Primjer 4. $f(x) = \dfrac{15+8x+x^2}{9-x^2}$

$\left.\begin{array}{l} 9 - x^2 \neq 0 \\ -x^2 \neq -9 \\ x^2 \neq 9 \\ x \neq \pm 3 \end{array}\right\} \Rightarrow D(f) = \mathbb{R} \setminus \{-3, 3\}$

$(15+8x+x^2):(x+3)$

$(x^2+8x+15):(x+3) = x+5$

$\underline{-x^2 \mp 3x}$

$5x+15$

$\underline{5x+15}$

0

$(9-x^2):(x+3)$

$(-x^2+9):(x-3) = -x-3$

$\underline{-x^2 \pm 3x}$

$-3x+9$

$\underline{-3x+9}$

0

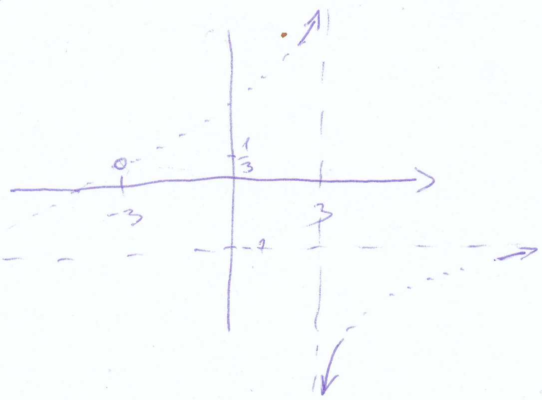
$\left.\begin{array}{l} \lim\limits_{x\to+\infty} \dfrac{15+8x+x^2}{9-x^2} = \dfrac{1}{-1} = -1 \\ \lim\limits_{x\to-\infty} \dfrac{15+8x+x^2}{9-x^2} = -1 \end{array}\right\}$ H.A. $\boxed{y=-1}$

$\lim\limits_{x\to3_+} \dfrac{15+8x+x^2}{9-x^2} = \left[\dfrac{15+24+9}{9-9_+} = \dfrac{48}{0_-}\right] = -\infty$

$\lim\limits_{x\to3_-} \dfrac{15+8x+x^2}{9-x^2} = \left[\dfrac{48}{9-9_-} = \dfrac{48}{0_+}\right] = +\infty$

V.A. $\boxed{x=3}$ neomeđena odozgo i odozdo

$3_- \cdot 3_- = 9_-$



$\lim\limits_{x\to-3_+} \dfrac{(x+3)(x+5)}{(x+3)(x-3)} = \lim\limits_{x\to-3_+} \dfrac{x+5}{-x-3} = \dfrac{2}{+6} = +\dfrac{1}{3}$

$\lim\limits_{x\to-3_-} f(x) = \dots = \dfrac{1}{3}$

$f(-x) = \dots \neq f(x)$, $\neq -f(x)$

nije trig.

Primjer 5. $f(x) = 2x + \sqrt{x^2 + x}$

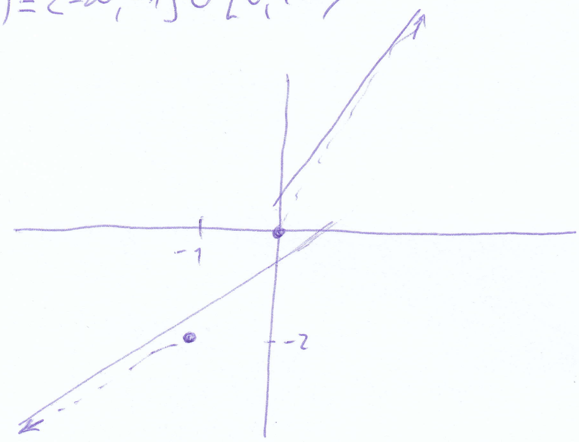
Vidi `http://lavica.fesb.hr/mat1/vjezbe/node108.html`

$\mathcal{D}(\sqrt{x}) = [0, +\infty\rangle \Rightarrow x^2 + x \geq 0$

$x(x+1) \geq 0$

$\Rightarrow \mathcal{D}(f) = \langle -\infty, -1] \cup [0, +\infty\rangle$

	$x=-1$		$x=0$	
x	$-$		$-$	$+$
$x+1$	$-$		$+$	$+$
$x(x+1)$	$+$		$-$	$+$



$$\lim_{x\to+\infty} (2x + \sqrt{x^2+x}) = +\infty$$

$$\lim_{x\to+\infty} \frac{2x+\sqrt{x^2+x}}{x} = \left[\frac{+\infty}{+\infty}\right] = \lim_{x\to+\infty} 2 + \sqrt{1+\frac{1}{x}} = 3$$

$$\Rightarrow \lim_{x\to+\infty} f(x) - 3x = \lim_{x\to+\infty} -x + \sqrt{x^2+x} = \lim_{x\to+\infty} \frac{\cancel{-x^2} + \cancel{x^2} + x}{x + \sqrt{x^2+x}} = \lim_{x\to+\infty} \frac{1}{1+\sqrt{1+\frac{1}{x}}} = \frac{1}{2}$$

$\Rightarrow$ K.A. $y(x) = 3x + \frac{1}{2}$

$$\lim_{x\to-\infty} (2x + \sqrt{x^2+x}) = \lim_{x\to+\infty} -2x + \sqrt{x^2-x} = [-\infty+\infty] = \lim_{x\to+\infty} \frac{-4x^2+x^2-x}{2x+\sqrt{x^2-x}} = \lim_{x\to+\infty} \frac{-3x^2-x}{2x+\sqrt{x^2-x}} = \lim_{x\to+\infty} \frac{-3-\frac{1}{x}}{\frac{2}{x}+\sqrt{\frac{1}{x^2}-\frac{1}{x^3}}} = \frac{-3}{0_+ + 0_+} = -\infty$$

$$\lim_{x\to-\infty} \frac{2x+\sqrt{x^2+x}}{x} = \lim_{x\to+\infty} \frac{-2x+\sqrt{x^2-x}}{-x} = +2-1 = 1$$

$$\lim_{x\to-\infty} f(x) - x = \lim_{x\to-\infty} x + \sqrt{x^2+x} = \lim_{x\to+\infty} -x + \sqrt{x^2-x} =$$

$$= \lim_{x\to+\infty} \frac{\cancel{-x^2} + \cancel{x^2} - x}{x + \sqrt{x^2-x}} = \frac{-1}{2} \quad y(x) = x - \frac{1}{2} \text{ K.A.}$$

$$\lim_{x\to-1} f(x) = -2 = f(-1), \quad \lim_{x\to 0} f(x) = 0 = f(0)$$

Primjer 6. $f(x) = \sqrt{4+x} - \sqrt{4-x}$

$4 + x \geq 0 \Rightarrow x \geq -4$

$4 - x \geq 0 \Rightarrow -x \geq -4 \;/\cdot(-1)$

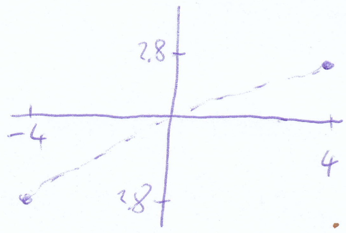
$x \leq 4$

$\Rightarrow \mathcal{D}(f) = [-4, 4]$

$$\lim_{x\to 4_-} f(x) = f(4) = \sqrt{8} = 2\sqrt{2} \approx 2.8$$

$$\lim_{x\to -4_+} f(x) = f(-4) = -\sqrt{8} = -2\sqrt{2} \approx -2.8$$

NEMA V.A.



NEMA HOR. I KOSIH ASIMPTOTA JER DOMENA NE UKLJUČUJE BESKONAČNOSTI

$\Rightarrow$ OGRANIČENA F-JA

$$f(-x) = \sqrt{4-x} - \sqrt{4+x} \neq f(x)$$

$$= -f(x) = -\sqrt{4+x} + \sqrt{4-x}$$

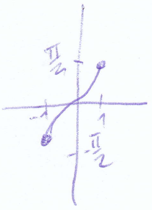
$\Rightarrow$ NEPARNA

NIJE PERIODIČNA

Primjer 7. $f(x) = \arcsin\left(\ln\left(x^2 - 4\right)\right)$

$\mathcal{D}(\arcsin) = [-1, 1]$

$-1 \leq \ln(x^2-4) \leq 1$



još $\mathcal{D}(\ln) = \langle 0, +\infty \rangle$

$x^2 - 4 > 0$

$x^2 > 4$

$x^2 \in \langle -\infty, -2 \rangle \cup \langle 2, +\infty \rangle$ ✓

$\ln(x^2-4) \leq 1 \quad / e^x$ rastuća

$x^2 - 4 \leq e$

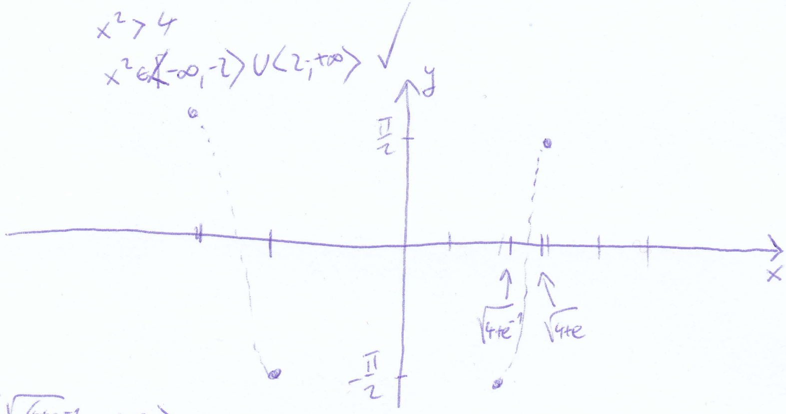
$x^2 \leq e + 4$

$-\sqrt{e+4} \leq x \leq \sqrt{e+4}$

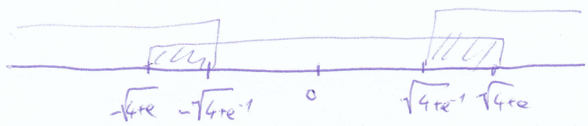
$-1 \leq \ln(x^2-4) \quad / e^x$ rastuća

$e^{-1} \leq x^2 - 4$

$4 + e^{-1} \leq x^2 \Rightarrow x \in \langle -\infty, -\sqrt{4+e^{-1}}] \cup [\sqrt{4+e^{-1}}, +\infty \rangle$



$\Rightarrow$



$f(-x) = \arcsin\left(\ln\left((-x)^2 - 4\right)\right) = f(x)$

PARNA

$\Rightarrow \mathcal{D}(f) = [-\sqrt{4+e}, -\sqrt{4+e^{-1}}] \cup [\sqrt{4+e^{-1}}, \sqrt{4+e}]$ ograničena

NIJE PERIODIČNA JER NE SADRŽI TRIG. FUNKCIJE.

$f(-x) = \_\_ = f(x) \Rightarrow$ PARNA

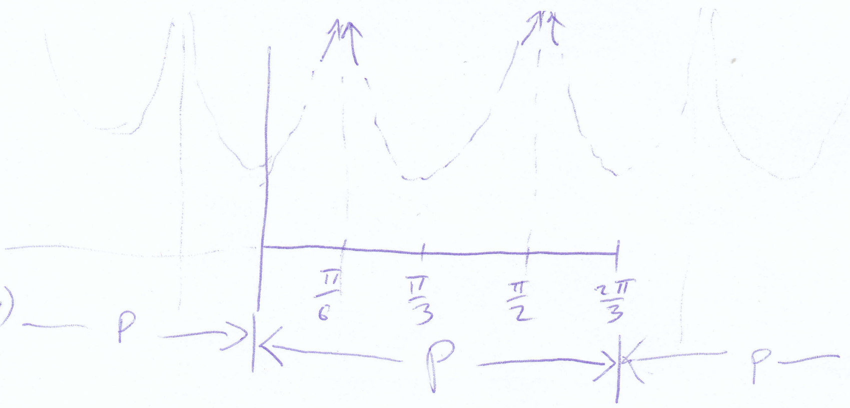
$\lim_{x \to \sqrt{4+e}} f(x) = f(\sqrt{4+e}) = f(-\sqrt{4+e}) = \arcsin(\ln 4 + e - 4) = \arcsin \ln e = \arcsin 1 = \frac{\pi}{2}$

$f(\sqrt{4+e^{-1}}) = f(-\sqrt{4+e^{-1}}) = \arcsin(\ln 4 + e^{-1} - 4) = \arcsin \ln e^{-1} = \arcsin(-1) = -\frac{\pi}{2}$

Primjer 8. $f(x) = \dfrac{1}{\cos^2(3x)}$

$\cos(x)$ je periodična sa $P = 2\pi$

$\cos(3x)$ — // — sa $P = \frac{2\pi}{3}$



$\cos(x + 2\pi) = \cos x$

$\cos(3(x+P)) = \cos(3x)$

$\cos(3x + 3P) = \cos(3x) = \cos(3x + 2\pi)$

$\Rightarrow 3P = 2\pi$

$P = \frac{2\pi}{3}$

ZAPRAVO SE MOŽE POKAZATI ZBOG $\cos^2(x)$ IMA PERIOD $P = \pi$ DA $f(x)$ IMA PERIOD $P = \frac{\pi}{3}$.

Promatramo između $x = 0$ i $x = \frac{2\pi}{3}$

$\mathcal{D}(f) = ?$

$\Rightarrow \mathcal{D}(f) = \mathbb{R} \setminus \left\{ \frac{k\pi}{3} + \frac{\pi}{6} : k \in \mathbb{Z} \right\}$

$\cos^2(3x)$ je u nazivniku

$\Rightarrow \cos^2(3x) \neq 0$

$\cos(3x) \neq 0$

$\left. \begin{array}{l} \lim_{x \to \frac{\pi}{6}} f(x) = \frac{1}{0_+} = +\infty \\ \lim_{x \to \frac{\pi}{2}} f(x) = +\infty \end{array} \right\}$ V.A. ZA $x = \frac{k\pi}{3} + \frac{\pi}{6}, \forall k \in \mathbb{Z}$

$\cos x = 0$ za $x = k\pi + \frac{\pi}{2}$

$\Rightarrow \cos 3x = 0$ za $3x = k\pi + \frac{\pi}{2}$

$x = \frac{k\pi}{3} + \frac{\pi}{6}$

$x_1 = \frac{\pi}{6}$

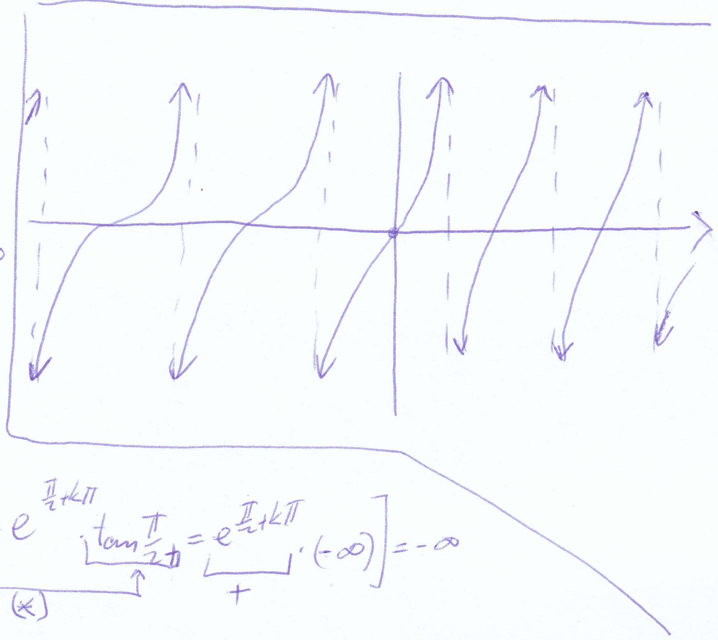
$x_2 = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$

neomeđena

$f(-x) = f(x)$ parna

Primjer 9. $f(x) = e^x \tan x$

tan periodična s osnovnim periodom $\pi \Rightarrow \tan x = \tan(x + k\pi)$ (*)



$\mathcal{D}(e^x) = \mathbb{R}$

$\mathcal{D}(\tan x) = \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$

$\Rightarrow \mathcal{D}(f) = \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$

$\lim\limits_{x \to \frac{\pi}{2}+} f(x) = \lim\limits_{x \to \frac{\pi}{2}+} e^x \tan x = \Big[\underbrace{e^{\frac{\pi}{2}}}_{+} \cdot (-\infty)\Big] = -\infty$

$\lim\limits_{x \to \frac{\pi}{2}-} f(x) = \lim\limits_{x \to \frac{\pi}{2}-} e^x \tan x = \Big[\underbrace{e^{\frac{\pi}{2}}}_{+} \cdot (+\infty)\Big] = +\infty$

$\lim\limits_{x \to (\frac{\pi}{2}+k\pi)_+} f(x) = \lim\limits_{x \to (\frac{\pi}{2}+k\pi)_+} e^x \tan x = \Big[e^{\frac{\pi}{2}+k\pi} \cdot \tan(\frac{\pi}{2}+k\pi)_+ = e^{\frac{\pi}{2}+k\pi} \cdot \tan\frac{\pi}{2}_+ = \underbrace{e^{\frac{\pi}{2}+k\pi}}_{+} \cdot (-\infty)\Big] = -\infty$ (*)

$\lim\limits_{x \to (\frac{\pi}{2}+k\pi)_-} f(x) = \lim\limits_{x \to (\frac{\pi}{2}+k\pi)_-} e^x \tan x = \Big[e^{\frac{\pi}{2}+k\pi} \cdot \tan(\frac{\pi}{2}+k\pi)_- = e^{\frac{\pi}{2}+k\pi} \cdot \tan(\frac{\pi}{2}_-) = \underbrace{e^{\frac{\pi}{2}+k\pi}}_{+} \cdot (+\infty)\Big] = +\infty$ (*)

NITI PARNA, NITI NEPARNA

$f(-x) = e^{-x} \tan(-x)$

$= e^{-x} \cdot -\tan x$

$= -e^{-x} \cdot \tan x \neq f(x)$

$\neq -f(x)$

$\Rightarrow$ NEOMEĐENA, NEOGRANIČENA

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} e^x \tan x = N/P$ $\quad \lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} e^x \cdot \tan x = N/P$ NEMA H.A.

Primjer 10. $f(x) = x\,e^{\frac{1}{x^2-1}}$ *Vidi* `http://lavica.fesb.hr/mat1/vjezbe/node109.html`

$\mathcal{D}(f) = \mathbb{R} \setminus \{-1, 1\}$

$= \langle -\infty, -1\rangle \cup \langle -1, 1\rangle \cup \langle 1, +\infty\rangle$

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} x \cdot e^{\frac{1}{x^2-1}} = \Big[+\infty \cdot e^{\frac{1}{+\infty}} = +\infty \cdot \underbrace{e^0}_{=1}\Big] = +\infty$ NEMA D.H.A.

$\lim\limits_{x \to +\infty} \frac{f(x)}{x} = \lim\limits_{x \to +\infty} e^{\frac{1}{x^2-1}} = \Big[e^{\frac{1}{+\infty}} = e^0\Big] = 1 \Rightarrow k = 1$ $\quad y(x) = x$ D.K.A.

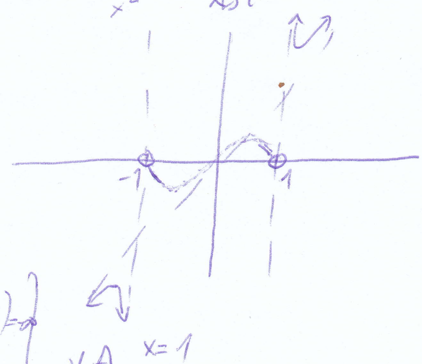
$\lim\limits_{x \to +\infty} (f(x) - x) = \lim\limits_{x \to +\infty} x\left(e^{\frac{1}{x^2-1}} - 1\right) = [+\infty \cdot 0_+] = \lim\limits_{x \to +\infty} \frac{e^{\frac{1}{x^2-1}} - 1}{\frac{1}{x}} = \left[\frac{0}{0}\right] \overset{L'H}{=} \lim\limits_{x \to +\infty} \frac{\frac{-2x}{(x^2-1)^2} \cdot e^{\frac{1}{x^2-1}}}{-\frac{1}{x^2}} = \lim\limits_{x \to +\infty} \frac{2x^3}{x^4 - 2x^2 + 1} \cdot e^{\frac{1}{x^2-1}} = 0 \cdot 1 = 0$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to +\infty} (-x) e^{\frac{1}{x^2-1}} = -\infty$ NEMA L.H.A.

$\lim\limits_{x \to -\infty} \frac{f(x)}{x} = \lim\limits_{x \to +\infty} e^{\frac{1}{x^2-1}} = 1 \Rightarrow k = 1$

$\lim\limits_{x \to -\infty} (f(x) - x) = \lim\limits_{x \to +\infty} -x\left(e^{\frac{1}{x^2-1}} - 1\right) = \ldots = 0$

$\Rightarrow$ L.K.A. $y(x) = x$



$\lim\limits_{x \to 1_+} f(x) = \lim\limits_{x \to 1_+} x\,e^{\frac{1}{x^2-1}} = 1 \cdot e^{\frac{1}{1_+ - 1}} = 1 \cdot e^{\frac{1}{0_+}} = 1 \cdot \underbrace{e^{+\infty}}_{=+\infty} = +\infty \Rightarrow \lim\limits_{x \to -1_-} f(x) = -\infty$

$\lim\limits_{x \to 1_-} f(x) = \lim\limits_{x \to 1_-} x\,e^{\frac{1}{x^2-1}} = 1 \cdot e^{\frac{1}{1_- - 1}} = 1 \cdot e^{\frac{1}{0_-}} = 1 \cdot \underbrace{e^{-\infty}}_{=0} = 0 = \lim\limits_{x \to -1_+} f(x)$

V.A. $x = 1$, $x = -1$

$f(-x) = -x\,e^{\frac{1}{x^2-1}} = -f(x) \Rightarrow$ NEPARNOST, NIJE PERIODIČNA

Nekoliko zadataka za samostalnu vježbu s rješenjem nalazi se u dodataku 5: 265-271 i 273.