Parcijalna integracija

Parcijalna integracija je primjena formule za derivaciju umnoška

$$f'(x)\cdot g(x)+f(x)\cdot g'(x)=(f(x)\cdot g(x))'$$

na integralni račun

$$\int f'(x)\cdot g(x)\,\mathrm{d}x+\int f(x)\cdot g'(x)\,\mathrm{d}x \;=\; \int (f(x)\cdot g(x))'\,\mathrm{d}x=f(x)\cdot g(x)+c$$

tako da se relativno komplicirani integral s lijeve strane pojednostavi na način

$$\begin{aligned}\int f'(x)\cdot g(x)\,\mathrm{d}x &= f(x)\cdot g(x)-\int f(x)\cdot g'(x)\,\mathrm{d}x,\\ \int_a^b f'(x)\cdot g(x)\,\mathrm{d}x &= [f(x)\cdot g(x)]_a^b-\int_a^b f(x)\cdot g'(x)\,\mathrm{d}x.\end{aligned}$$

Zamjenom varijabli $u=f(x)$ i $v=g(x)$ možemo pisati i ovako:

$$\begin{aligned}\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x &= uv-\int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x,\\ \int u\,\mathrm{d}v &= uv-\int v\,\mathrm{d}u\end{aligned}$$

Primjer. Koristeći zamjenu $u=x$ i $\mathrm{d}v=\sin x\,\mathrm{d}x$ izračunati

$$\int x\sin x\,\mathrm{d}x=\left\{\begin{array}{ll} u=x; & \mathrm{d}u=u'=1\,\mathrm{d}x \\ \mathrm{d}v=\sin x\,\mathrm{d}x; & v=\int \mathrm{d}v=\int\sin x\,\mathrm{d}x=-\cos x\end{array}\right\}=uv-\int v\,\mathrm{d}u=x\cdot(-\cos x)-\int(-\cos x)\cdot 1\,dx$$

$$=-x\cos x+\int\cos x\,dx=-x\cos x+\sin x+C$$

Presudan je odabir: u se treba pojednostaviti derivacijom i/ili $\mathrm{d}v$ pojednostaviti integracijom. Važno!

Primjer. $$\int_0^1 xe^x\,\mathrm{d}x=\left\{\begin{array}{ll} u=x & du=1dx \\ dv=e^x dx & v=\int dv=\int e^x dx=e^x\end{array}\right\}=uv-\int v du=xe^x-\int e^x dx=xe^x-e^x$$

$$\int x\,(x+1)^5\,\mathrm{d}x=\left\{\begin{array}{ll} u=x & du=dx \\ dv=(x+1)^5dx & v=\int dv=\int(x+1)^5dx=\frac{(x+1)^6}{6}\end{array}\right\}$$

$$=uv-\int v du$$

$$=x\cdot\frac{(x+1)^6}{6}-\int\frac{(x+1)^6}{6}dx$$

$$=\frac{x(x+1)^6}{6}-\frac{1}{6}\cdot\frac{(x+1)^7}{7}+C$$

$$\int\frac{\ln x}{x^2}\,\mathrm{d}x=\left\{\begin{array}{ll} u=\ln x; & du=\frac{1}{x}\cdot dx \\ dv=\frac{1}{x^2}dx; & v=\int dv=\int\frac{dx}{x^2}=-\frac{1}{x}\end{array}\right\}$$

$$=-\frac{\ln x}{x}-\int\left(-\frac{1}{x}\right)\cdot\left(\frac{1}{x}\right)dx$$

$$=-\frac{\ln x}{x}+\int\frac{1}{x^2}dx=-\frac{\ln x}{x}-\frac{1}{x}+C$$

$$\int_0^1 (x+2)\,(x+1)^8\,\mathrm{d}x=\left\{\begin{array}{ll} u=x+2 & du=dx \\ dv=(x+1)^8dx & v=\frac{(x+1)^9}{9}\end{array}\right\}=$$

$$=\left[(x+2)\frac{(x+1)^9}{9}\right]_0^1-\int_0^1\frac{(x+1)^9}{9}dx$$

$$=\left(\cancel{3}^{1}\cdot\frac{2^9}{\cancel{9}_3}-2\cdot\frac{1}{9}\right)-\frac{1}{9}\left(\frac{(x+1)^{10}}{10}\right)_0^1=\frac{2^9}{3}-\frac{2}{9}-\frac{1}{9}\left(\frac{2^{10}}{10}-\frac{1}{10}\right)$$

$$=\ldots$$

$$\int\frac{\ln(\sin x)}{\cos^2 x}\mathrm{d}x=\left\{\begin{array}{ll} u=\ln(\sin x) & du=\frac{\cos x}{\sin x}dx \\ dv=\frac{dx}{\cos^2 x} & v=\tan x=\frac{\sin x}{\cos x}\end{array}\right\}$$

$$=\tan x\ln(\sin x)-\int\frac{\cos x}{\sin x}\cdot\frac{\sin x}{\cos x}dx=\tan x\ln(\sin x)-x+C$$

$$\cancel{\int\arctan x\,\mathrm{d}x}=\int\arctan(x)\cdot 1\,\mathrm{d}x=\left\{\begin{array}{ll} u=\arctan x; & \mathrm{d}u=u'=\frac{\mathrm{d}x}{1+x^2} \\ \mathrm{d}v=1\,\mathrm{d}x; & v=\int\mathrm{d}v=\int 1\,\mathrm{d}x=x\end{array}\right\}=x\arctan x-\int\frac{x}{1+x^2}dx$$

$$[\ln(1+x^2)]'=\frac{2x}{1+x^2}$$

$$=x\arctan x-\frac{1}{2}\ln(1+x^2)+C$$

Zadaci za samostalan rad:

1. $\int x\cos x\,\mathrm{d}x$ 2. $\int 2x(x+1)^3\,\mathrm{d}x$ 3. $\int x^2\ln x\,\mathrm{d}x$ 4. $\int_1^2 2x\,e^x\,\mathrm{d}x$ 5. $\int(x-1)\sin(x-1)\,\mathrm{d}x$

6. $\int 3x\,e^{x+1}\,\mathrm{d}x$ 7. Riješiti parcijalnom integracijom korištenjem $u=\ln x$ i $\mathrm{d}v=1\,\mathrm{d}x$: $\int\ln x\,\mathrm{d}x$

Često je parcijalnu integraciju potrebno upotrijebiti nekoliko puta u nizu:

- Primjer. $\int x^3 e^x \, dx = \left\{ \begin{matrix} u = x^3 & du = 3x^2 dx \\ dv = e^x dx & v = e^x \end{matrix} \right\} = x^3 e^x - 3\int x^2 e^x dx = \left\{ \begin{matrix} u = x^2 & du = 2x dx \\ dv = e^x dx & v = e^x \end{matrix} \right\} =$

$= x^3 e^x - 3\left(x^2 e^x - 2\int x e^x dx\right) = x^3 e^x - 3x^2 e^x + 6\int x e^x dx = \left\{ \begin{matrix} u = x & du = dx \\ dv = e^x dx, & v = e^x \end{matrix} \right\} =$

$= x^3 e^x - 3x^2 e^x + 6\left(x e^x - \int e^x dx\right) = x^3 e^x - 3x^2 e^x + 6x e^x - e^x + C$

$\int_0^\pi 2x^2 \cos x \, dx = \left\{ \begin{matrix} u = 2x^2 & du = 4x\,dx \\ dv = \cos x\,dx & v = \sin x \end{matrix} \right\} = \left(2x^2 \sin x\right)\Big|_0^\pi - 4\int_0^\pi x \sin x\,dx = \left\{ \begin{matrix} u = x & du = dx \\ dv = \sin x\,dx & v = -\cos x \end{matrix} \right\} =$

$= \underbrace{\left(2\pi^2 \underbrace{\sin\pi}_{=0} - 2\cdot 0^2 \sin 0\right)}_{=0} - 4\left\{\left(-x\cos x\right)\Big|_0^\pi - \int_0^\pi (-\cos x)\,dx\right\} = +4\left(x\cos x\right)\Big|_0^\pi - 4\left(\sin x\right)\Big|_0^\pi = -4\pi$

X - $\int \sin^3 x \cos^5 x \, dx = \left\{ \begin{matrix} u = \sin^2 x & du = 2\sin x \cos x\,dx \\ dv = \cos^5 x \sin x\,dx & v = \int dv = -\frac{1}{6}\cos^6 x \end{matrix} \right\} = -\frac{1}{6}\sin^2 x \cos^6 x + \int \frac{1}{3}\cos^7 x \sin x\,dx$
$= -\frac{1}{6}\sin^2 x \cos^6 x - \frac{1}{3}\cdot\frac{1}{8}\cos^8 x + C$

» *Teži zadatak* «
Opcija je da iskoristiš $\sin^2 x = 1 - \cos^2 x$.

DRUGI NAČIN:

$\int \sin^3 x \cos^5 x\,dx = \int (1 - \cos^2 x)\sin x \cos^5 x\,dx = \int \cos^5 x \sin x\,dx - \int \cos^7 x \sin x\,dx = -\frac{1}{6}\cos^6 x + \frac{1}{8}\cos^8 x + C$

Samostalno riješiti: 1. $\int x^3 \sin x \, dx$, 2. $\int x^2 e^{-x} \, dx$, 3. $\int_0^1 x^2 (x+1)^4 \, dx$, 4. $\int_0^\pi \sin^5 x \cos^3 x \, dx$

Ponekad ~~se uzastopnom~~ parcijalnom integracijom naizgled »zavrtimo u krug«.

X Primjer. $\int_0^{\pi/2} e^x \cos x \, dx = \left\{ \begin{matrix} u=\cos x & du=-\sin x\,dx \\ dv=e^x dx & v=e^x \end{matrix} \right\} = \left(e^x\cos x\right)\Big|_0^{\frac{\pi}{2}} + \int_0^{\pi/2} e^x \sin x\,dx = \left\{ \begin{matrix} u=\sin x & du=\cos x\,dx \\ dv=e^x dx & v=e^x \end{matrix} \right\}$

$= \left(e^{\frac{\pi}{2}}\underbrace{\cos\frac{\pi}{2}}_{=0} - e^0\cos 0\right) + \underbrace{\left(e^x \sin x\right)\Big|_0^{\frac{\pi}{2}}}_{=\left(e^{\frac{\pi}{2}}\sin\frac{\pi}{2} - e^0\sin 0\right)} - \int_0^{\frac{\pi}{2}} e^x\cos x\,dx = -1 + e^{\frac{\pi}{2}} - \int_0^{\pi/2} e^x\cos x\,dx$ POJAVIO SE POČETNI INTEGRAL

DAKLE: $\boxed{\int_0^{\pi/2} e^x\cos x\,dx = -1+e^{\frac{\pi}{2}} - \int_0^{\pi/2} e^x \cos x\,dx} \Rightarrow \boxed{2\int_0^{\pi/2} e^x\cos x\,dx = -1+e^{\frac{\pi}{2}}} \Rightarrow \int_0^{\pi/2} e^x \cos x\,dx = \frac{-1+e^{\frac{\pi}{2}}}{2}$

X $\int_0^{\pi} \sin(x)\, e^x \, dx =$ Samostalno.

$\int \sin^2 x \, dx = \int \sin x \cdot \sin x\,dx = \left\{ \begin{matrix} u=\sin x & du=\cos x\,dx \\ dv=\sin x\,dx & v=-\cos x \end{matrix} \right\} = -\sin x\cos x + \int \cos^2 x\,dx =$ $\boxed{\cos^2 x + \sin^2 x = 1}$ $\cos^2 x = 1-\sin^2 x$

$= -\sin x\cos x + \int (1-\sin^2 x)\,dx$

$= -\sin x \cos x + \int dx - \int \sin^2 x\,dx$

$= -\sin x\cos x + x + C - \int \sin^2 x\,dx$ POJAVIO SE POČETNI INTEGRAL

$\Rightarrow \boxed{\int \sin^2 x\,dx = -\sin x\cos x + x + C - \int \sin^2 x\,dx}$

$\Rightarrow \underbrace{\int \sin^2 x\,dx + \int \sin^2 x\,dx}_{=2\int \sin^2 x\,dx} = -\sin x \cos x + x + C \Rightarrow \int \sin^2 x\,dx = \frac{-\sin x\cos x + x + C}{2}$

$\int \cos^2 x \, dx =$ Samostalno.

Integriranje zamjenom varijabli - supstitucijom

Supstitucija se sastoji u tome da se nekom dopustivom zamjenom integracijske varijable ili podintegralnog izraza polazni integral svede na neke od onih tabličnih. (*Vidi teorem 4.2.5 iz knjige*)

Primijenimo pravilo za deriviranje kompozicije funkcije:

$$F'(g(x)) \cdot g'(x) = (F \circ g)'(x).$$

Integrirajmo:

$$\int F'(g(x)) \cdot g'(x) = \int (F \circ g)'(x) = (F \circ g)(x) + c$$

Zaključimo: ako je f neprekidna funkcija i F njena primitivna funkcija (*čitaj:* $F'(x) = f(x)$), a uz to g neprekidno derivabilna funkcija (*čitaj:* $g'(x)$ neprekidna). Tada je

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x \quad = \quad F(g(x)) + c.$$

Drugim riječima, napravili smo zamjenu varijabli $t = g(x)$:

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = \left\{ \begin{array}{c} t = g(x) \\ \mathrm{d}t = \mathrm{d}\,(g(x)) = g'(x)\,\mathrm{d}x \end{array} \right\} = \int f(t)\,\mathrm{d}t = F(t) + c = F(g(x)) + c.$$

Primjer. Iskoristiti supstituciju varijabli $2x + 1 = u$ za rješenje integrala $\int \underbrace{(2x+1)}_{u}^2\,\mathrm{d}x. = \left\{ \begin{array}{l} u = 2x+1 \\ du = 2dx \end{array}, \; dx = \frac{du}{2} \right\}$

$$= \int u^2 \frac{du}{2} = \frac{1}{2}\int u^2 du = \frac{1}{2}\frac{u^3}{3} + C = \frac{1}{2}\cdot\frac{1}{3}(2x+1)^3 + C = \frac{1}{6}(2x+1)^3 + C$$

Riješiti: $\int_0^1 (2x+1)^2\,\mathrm{d}x = \left\{ \begin{array}{l} u = 2x+1 \\ du = 2dx, \; dx = \frac{1}{2}du \end{array} \right\} =$

$$= \int_{2\cdot 0+1}^{2\cdot 1+1} u^2 \cdot \frac{1}{2} du = \frac{1}{2}\int_1^3 u^2 du = \frac{1}{2}\left(\frac{u^3}{3}\right)\Big|_1^3 = \frac{1}{2}\cdot\frac{27-1}{3} = \frac{13}{3}$$

Pazi na granice integracije! Možemo prvo riješiti neodređeni integral i na kraju uvrstiti granice integracije.

Iskoristiti supstituciju varijabli $x^2 = u$ za rješenje integrala $\int 2x \cos(x^2)\,\mathrm{d}x$

$$\int 2x \cos(\underbrace{x^2}_{u}) dx = \left\{ \begin{array}{l} u = x^2 \\ du = 2x dx \end{array} \right\} = \int \cos u \, du = \sin u + C = \sin x^2 + C$$

PRVI NAČIN: PRI SUPSTITUCIJI ZAMIJENI ODMAH GRANICE INTEGRACIJE

$$\int_{-1}^{1} 2x \cos(x^2)\,\mathrm{d}x = \left\{ \begin{array}{l} u = x^2 \\ du = 2x dx \end{array} \right\} = \int_{(-1)^2}^{1^2} \cos u \, du = \int_1^1 \cos u \, du = (\sin u)\Big|_1^1 = 0$$

Pazi na granice integracije!

DRUGI NAČIN: ZABORAVI NA GRANICE INTEGRACIJE DOK NE RIJEŠIŠ NEODREĐENI INTEGRAL

$$\int_{-1}^{1} 2x\cos(x^2) dx = \left\{ \begin{array}{l} u = x^2 \\ du = 2x dx \end{array} \right\} = \int \cos u \, du = \sin u = \sin(x^2)\Big|_{-1}^{1} = \sin 1^2 - \sin(-1)^2 = \sin 1 - \sin 1 = 0$$

"ZABORAVILI" SMO NA GRANICE INTEGRACIJE DOK SE NISMO VRATILI NA ORIGINALNU VARIJABLU "x"

OZNAKA:

$$\sin(x^2)\Big|_a^b = \left(\sin(x^2)\right)_a^b = \sin(b^2) - \sin(a^2)$$

Zadaci:

- $\int \cos(5t)\,dt = \left\{ \begin{array}{l} u=5t \\ du=5dt \Rightarrow dt=\frac{du}{5} \end{array} \right\} = \int \cos u \cdot \frac{1}{5} du$

$= \frac{1}{5}\sin u + C = \frac{1}{5}\sin(5t) + C$

$$\boxed{\begin{array}{l} \text{PROVJERA:} \\ \left(\frac{1}{5}\sin(5t)+C\right)' = \frac{1}{5}\cdot\cos(5t)\cdot 5 \\ \qquad = \cos(5t) \end{array}}$$

$\int (4y-9)^{-\frac{1}{3}}\,dy = \left\{ \begin{array}{l} u=4y-9 \\ du=4dy \Rightarrow dy=\frac{1}{4}du \end{array} \right\}$

$= \frac{1}{4}\int u^{-\frac{1}{3}}du = \frac{1}{4}\cdot\frac{u^{\frac{2}{3}}}{\frac{2}{3}} = \frac{3}{8}u^{\frac{2}{3}} + C$

$= \frac{3}{8}(4y-9)^{\frac{2}{3}} + C$

$\int x^2 (\underbrace{x^3-8}_{u})^{11}\,dx = \left\{ \begin{array}{l} u=x^3-8 \\ du=3x^2dx \Rightarrow x^2dx=\frac{du}{3} \end{array} \right\}$

$= \frac{1}{3}\int u^{11}du = \frac{1}{3}\cdot\frac{1}{12}\cdot u^{12} + C = \frac{1}{36}(x^3-8)^{12} + C$

- $\int \frac{\overbrace{2x\,dx}^{du}}{\underbrace{x^2-4}_{u}} = \left\{ \begin{array}{l} u=x^2-4 \\ du=2x\,dx \end{array} \right\} = \int\frac{du}{u} = \ln|u| + C = \ln|x^2-4| + C$

$$\boxed{\begin{array}{l} \text{PROVJERA:} \\ \left(\ln|x^2-4|\right)' = \frac{1}{x^2-4}\cdot 2x = \frac{2x}{x^2-4} \end{array}}$$

$\int_0^1 \frac{2x}{x^2-4}\,dx = \left(\ln|x^2-4|\right)\Big|_0^1 = \ln|-3| - \ln|-4| = \ln 3 - \ln 4$ Pazi na domenu!

- $\int_1^2 x\cos(\underbrace{3x^2+4}_{u})\,dx = \left\{ \begin{array}{l} u=3x^2+4 \\ du=3\cdot 2x\,dx,\ x\,dx=\frac{1}{6}du \end{array} \right\} = \int_7^{16} \frac{1}{6}\cos u\,du = \frac{1}{6}\left[\sin u\right]_7^{16} = \frac{1}{6}(\sin 16 - \sin 7)$

$\int \frac{\sin(\overbrace{\ln 3x}^{u})}{x}\,dx = \left\{ \begin{array}{l} u=\ln(3x) \\ du=\frac{1}{3x}\cdot 3\,dx = \frac{1}{x}dx \end{array} \right\} = \int \sin u\,du$

$= -\cos u + C = -\cos(\ln(3x)) + C$

$\int \frac{2x+1}{\underbrace{x^2+2x+2}_{u}}\,dx = \left\{ \begin{array}{l} u=x^2+2x+2 \\ du=(2x+2)dx \end{array} \right\} = \int\frac{2x+2}{x^2+2x+2}dx - \int\frac{dx}{x^2+2x+2} = \underline{\ln|x^2+2x+2|} - \underline{\arctan(x+1)} + C$

$\int \frac{2x+2}{x^2+2x+2}dx \quad \left\{ \begin{array}{l} u=x^2+2x+2 \\ du=(2x+2)dx \end{array} \right\} = \int\frac{du}{u} = \ln|u| = \ln|x^2+2x+2| + C$

$\int\frac{dx}{x^2+2x+2} = \int\frac{dx}{x^2+2x+1+1} = \int\frac{dx}{(x+1)^2+1} = \left\{ \begin{array}{l} u=x+1 \\ du=dx \end{array} \right\} = \int\frac{du}{u^2+1} =$

$= \arctan u = \arctan(x+1) + C$

$\int \tan x\,dx = \int\frac{\sin x}{\cos x}dx = \left\{ \begin{array}{l} u=\cos x \\ du=-\sin x\,dx \end{array} \right\}$

$= \int\frac{-du}{u} = -\ln|u| = -\ln|\cos x| + C$

$$\boxed{\begin{array}{l} \text{PROVJERA:} \\ \left(-\ln|\cos x|\right)' = -\frac{1}{\cos x}\cdot(-\sin x) = \tan x \end{array}}$$

$\int_2^3 \frac{2x^2+x+2}{x^2-1}\,dx = \int_2^3\left(2 + \frac{x+4}{x^2-1}\right)dx$

$(2x^2+x+2) : (x^2-1) = 2$
$-(2x^2-2)$
$\overline{\quad x+4}$

$= \underbrace{\int_2^3 2\,dx}_{=2} + \underbrace{\int_2^3\frac{x+4}{x^2-1}dx}_{=\frac{1}{2}(\ln 8-\ln 3)+2(\ln\frac{1}{2}-\ln\frac{1}{3})}$

$= 2 + \frac{1}{2}(\ln 8 - \ln 3) + 2\left(\ln\frac{1}{2} - \ln\frac{1}{3}\right)$

$\int_2^3 2\,dx = (2x)\Big|_2^3 = 6-4 = 2$

$\int_2^3\frac{x+4}{x^2-1}dx = \int_2^3\frac{x}{x^2-1}dx + \int_2^3\frac{4}{x^2-1}dx =$

$\int_2^3\frac{x\,dx}{x^2-1} = \left\{ \begin{array}{l} u=x^2-1 \\ du=2x\,dx \end{array} \right\} = \frac{1}{2}\int_3^8\frac{du}{u} = \frac{1}{2}\left(\ln|u|\right)\Big|_3^8 = \frac{1}{2}(\ln 8 - \ln 3)$

$\int_2^3\frac{4}{x^2-1}dx = \frac{4}{2}\left(\ln\left|\frac{x-1}{x+1}\right|\right)\Big|_2^3 = 2\left(\ln\frac{1}{2} - \ln\frac{1}{3}\right)$