Seminar 12 (Derivacija funkcije)

Potrebno predznanje

Derivacija funkcije f u točki x_0 odgovara vrijednosti sljedećeg limesa: $\lim\limits_{x \to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$. Kada promatramo taj izraz kao funkciju s parametrom x_0 govorimo o funkciji koju nazivamo derivacija funkcije f, odnosno f' (čitaj: »f crtica«). Možemo pisati, $f'(x) = \lim\limits_{y \to x} \dfrac{f(y)-f(x)}{y-x}$.

Derivacije osnovnih funkcija računate su na predavanjima i u knjizi. Derivacije osnovnih funkcija sadržane su u sljedećoj tablici (opširniju tablicu možete naći na `http://hr.wikipedia.org/wiki/Tablica_integrala`):

f	$\frac{df}{dx}$
konstanta	0
x	1
x^2	$2x$
$\frac{1}{x}$	$-\frac{1}{x^2}$
x^α $(\alpha \neq 0)$	$\alpha x^{\alpha-1}$

f	$\frac{df}{dx}$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \ln a}$
e^x	e^x
a^x	$a^x \ln a$

f	$\frac{df}{dx}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\cot x$	$\frac{-1}{\sin^2 x}$

f	$\frac{df}{dx}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\text{arccot}\, x$	$-\frac{1}{1+x^2}$

Pravila za deriviranje osnovnih računskih operacija:

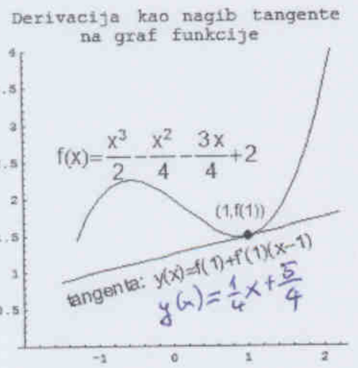
$$\begin{aligned}
(f \pm g)' &= f' \pm g' \\
(f \cdot g)' &= f' \cdot g + f \cdot g' \\
(c \cdot f)' &= c \cdot f' \qquad (c\,\text{konstanta}) \\
\left(\frac{f}{g}\right)' &= \frac{f' \cdot g - f \cdot g'}{g^2}
\end{aligned}$$

Pravilo za deriviranje kompozicije funkcija:

$$f(g(x))' = f'(g(x)) \cdot g'(x)$$

Više derivacije računamo uzastopnim deriviranjem odgovarajući broj puta.

Derivacija funkcije f u nekoj točki odgovara nagibu tangente grafa u toj točki, kao što je nacrtano na na sljedećoj slici:



$$\begin{aligned}
f'(x) &= \left(\frac{x^3}{2} - \frac{x^2}{4} - \frac{3x}{4} + 2\right)' \\
&= \left(\frac{x^3}{2}\right)' - \left(\frac{x^2}{4}\right)' - \left(\frac{3x}{4}\right)' + 2' \\
&= \frac{1}{2}(x^3)' - \frac{1}{4}(x^2)' - \frac{3}{4}(x)' + 0 \\
&= \frac{1}{2} \cdot 3x^2 - \frac{1}{4} \cdot 2x - \frac{3}{4} \cdot 1 + 0
\end{aligned}$$

$$\Rightarrow f'(x) = \frac{3}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

$$f(1) = \frac{1}{2} - \frac{1}{4} - \frac{3}{4} + 2 = \frac{3}{2}$$

$$f'(1) = \frac{3}{2} - \frac{1}{2} - \frac{3}{4} = \frac{1}{4}$$

TANGENTA KROZ $(1, f(1))$ GRAFA:

$$y(x) = \frac{3}{2} + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{5}{4}$$

Jednadžba tangente na graf funkcije f u točki $T(x_0, f(x_0))$ jest $y(x) = f(x_0) + f'(x_0) \cdot (x - x_0)$.

L'Hopitalovo pravilo: U slučaju neodređenog oblika limesa $\lim \frac{f(x)}{g(x)}$, vrijedi $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$.

Glavni dio seminara

Primjer 1. Odrediti tangentu na funkciju $f(x) = \frac{x^3}{2} - \frac{x^2}{4} - \frac{3x}{4} + 2$ tamo gdje je $x = 1$. Vidi sliku na strani ispred.

VIDI PRETHODNU STRANU

PRIMJER 2. DRUGI NAČIN

$$f(x) = \frac{2}{\sqrt[3]{(x-1)^2}} = 2 \cdot (x-1)^{-\frac{2}{3}}$$

$$f'(x) = \left(2 \cdot (x-1)^{-\frac{2}{3}}\right)' = 2 \cdot \underbrace{\left((x-1)^{-\frac{2}{3}}\right)'}_{(f(g(x)))'} = \left\{ \begin{array}{l} g(x) = x-1, \ g'(x) = 1 \\ f(x) = x^{-\frac{2}{3}}, \ f'(x) = -\frac{2}{3} x^{-\frac{5}{3}} \\ \cancel{f'(g(x)) = -\frac{2}{3}(g(x))} \end{array} \right\}$$

$$f'(g(x)) = -\frac{2}{3}(g(x))^{-\frac{5}{3}}$$

$$f'(x-1) = -\frac{2}{3}(x-1)^{-\frac{5}{3}}$$

$$= 2 \cdot \underbrace{\left(-\frac{2}{3}\right)(x-1)^{-\frac{5}{3}}}_{f'(g(x))} \cdot \underbrace{1}_{g'(x)} = -\frac{4}{3} \frac{1}{\sqrt[3]{(x-1)^5}}$$

ISTI REZULTAT

Derivirati funkcije sa prošlog seminara:

Primjer 2. $f(x) = \frac{2}{\sqrt[3]{(x-1)^2}}$

PRVI NAČIN

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$\left(\frac{2}{\sqrt[3]{(x-1)^2}}\right)' = \frac{2' \cdot \sqrt[3]{(x-1)^2} - 2 \cdot \left(\sqrt[3]{(x-1)^2}\right)'}{\sqrt[3]{(x-1)^4}} = (*) = \frac{0 \cdot \sqrt[3]{(x-1)^2} - 2 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt[3]{x-1}}}{\sqrt[3]{(x-1)^4}} = \frac{-\frac{4}{3} \cdot \frac{1}{\sqrt[3]{x-1}}}{\frac{\sqrt[3]{(x-1)^4}}{1}} = -\frac{4}{3} \cdot \frac{1}{\sqrt[3]{(x-1)^5}}$$

$$(*): \left(\sqrt[3]{(x-1)^2}\right)' = \underbrace{\left((x-1)^{\frac{2}{3}}\right)'}_{(f(g(x)))' = f'(g(x)) \cdot g'(x)} = \left\{ \begin{array}{l} g(x) = x-1, \ g'(x) = 1 \\ f(x) = x^{\frac{2}{3}}, \ f'(x) = \frac{2}{3} x^{-\frac{1}{3}} \end{array} \right\} = \underbrace{\frac{2}{3}(g(x))^{-\frac{1}{3}}}_{f'(g(x))} \cdot \underbrace{1}_{g'(x)} = \frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{2}{3} \frac{1}{\sqrt[3]{x-1}}$$

Primjer 3. $f(x) = \frac{x}{x-1}$

$$f'(x) = \left(\frac{x}{x-1}\right)' = \frac{x' \cdot (x-1) - x \cdot (x-1)'}{(x-1)^2} = \frac{1 \cdot (x-1) - x \cdot 1}{(x-1)^2} = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

Primjer 4. $f(x) = \frac{x^2+1}{x^2-1}$ *Vidi* `http://lavica.fesb.hr/mat1/vjezbe/node107.html`

$$f'(x) = \left(\frac{x^2+1}{x^2-1}\right)' = \frac{(x^2+1)' \cdot (x^2-1) - (x^2+1) \cdot (x^2-1)'}{(x^2-1)^2} = \frac{2x \cdot (x^2-1) - (x^2+1) \cdot 2x}{(x^2-1)^2} =$$

$$= \frac{2x \cdot (\cancel{x^2} - 1 - \cancel{x^2} - 1)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

Primjer 5. Usporedi derivaciju funkcije $f_1(x) = 2 \sin x \cos x$ sa $f_2(x) = \sin(2x)$

$$f_1'(x) = 2\left[(\sin x)' \cos x + \sin x \cdot (\cos x)'\right] = 2(\cos^2 x - \sin^2 x)$$

$$f_2'(x) = \underbrace{\left[\sin(2x)\right]'}_{\left[f(g(x))\right]'} = \left\{ \begin{array}{ll} f = \sin x & f' = \cos x \\ g = 2x & g' = 2 \end{array} \right\} = 2\cos(2x)$$

$$= f'(g(x)) \cdot g'(x)$$

VIDI TRIGONOMETRIJSKE FORMULE: VRIJEDI $\sin(2x) = 2 \sin x \cos x \Rightarrow f_1 = f_2$

$\cos(2x) = \cos^2 x - \sin^2 x \Rightarrow f_1' = f_2'$

L'Hopitalovim pravilom izračunati sljedeće:

Primjer 6. $\lim\limits_{x\to -3} \frac{15+8x+x^2}{9-x^2} = \left[\frac{15-24+9}{9-9}\right] = \left[\frac{0}{0}\right] \overset{L'H}{=} \lim\limits_{x\to -3} \frac{(15+8x+x^2)'}{(9-x^2)'} = \lim\limits_{x\to -3} \frac{8+2x}{-2x} = \frac{8-6}{6} = \frac{1}{3}$

L'HOPITALOVO PRAVILO SE MOŽE PRIMIJENITI SAMO KAD JE LIMES NEODREĐENOG OBLIKA $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$

Primjer 7. $\lim\limits_{x\to 1} \frac{\sqrt{2-x^2}-x}{x-1} = \left[\frac{\sqrt{2-1}-1}{1-1}\right] = \left[\frac{0}{0}\right] \overset{L'H}{=} \lim\limits_{x\to 1} \frac{(\sqrt{2-x^2}-x)'}{(x-1)'} = \lim\limits_{x\to 1} \frac{\frac{-2x}{2\sqrt{2-x^2}}-1}{1} = \frac{\frac{-2}{2\cdot 1}-1}{1} = \frac{-2}{1} = -2$

Vidi primjer 9. iz seminara 10!

$\left(\sqrt{2-x^2}\right)' = \begin{cases} g(x) = 2-x^2, & g'(x) = -2x \\ f(x) = \sqrt{x}, & f'(x) = \frac{1}{2\sqrt{x}} \end{cases} = \frac{1}{2\sqrt{2-x^2}}\cdot(-2x)$

$(f(g(x)))' \qquad f'(g(x))\cdot g'(x)$

$\Rightarrow \lim\limits_{x\to 1} \frac{\sqrt{2-x^2}-x}{x-1} = -2$

Primjer 8. $\lim\limits_{x\to 1} \frac{x-1}{\sqrt[3]{x}-1} = \left[\frac{0}{0}\right] \overset{L'H}{=} \lim\limits_{x\to 1} \frac{(x-1)'}{(\sqrt[3]{x}-1)'} = \lim\limits_{x\to 1} \frac{1}{\frac{1}{3\sqrt[3]{x^2}}} = \frac{1}{\frac{1}{3}} = 3$

Vidi primjer 10. iz seminara 10!

Primjer 9. $\lim\limits_{x\to 0+} x^x = [0_+^{0+}] =$

{ L'HOPITALOVO PRAVILO MOŽEMO KORISTITI SAMO KOD RAZLOMKA }

POKUŠAVAMO "IZMISLITI" RAZLOMAK

$a = e^{\ln a}$ $\quad \ln a^b = b \ln a$

$\Rightarrow x^x = e^{\ln x^x} = e^{x\ln x} = e^{\frac{\ln x}{\frac{1}{x}}}$

$\lim\limits_{x\to 0+} e^{\frac{\ln x}{\frac{1}{x}}} = \left[e^{\frac{-\infty}{+\infty}}\right] = e^{\lim\limits_{x\to 0+}\frac{\ln x}{\frac{1}{x}}} = e^0 = 1$

$\lim\limits_{x\to 0+} \frac{\ln x}{\frac{1}{x}} \overset{L'H}{=} \lim\limits_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0+} \frac{-x^{\cancel{2}}}{\cancel{x}} = \lim\limits_{x\to 0+} -x = 0$

Izračunati naznačene više derivacije funkcija:

Primjer 10. $f(x) = \log x$, $f^{(4)}(2) = ?$

$f'(x) = (\log_{10} x)' = \frac{1}{x\ln 10}$

$f''(x) = \left(\frac{1}{x\ln 10}\right)' = \frac{1}{\ln 10}\cdot\left(\frac{1}{x}\right)' = \frac{1}{\ln 10}\cdot\left(-\frac{1}{x^2}\right)$

$f'''(x) = \left(-\frac{1}{\ln 10\, x^2}\right)' = -\frac{1}{\ln 10}\cdot\left(\frac{1}{x^2}\right)' = -\frac{1}{\ln 10}\cdot\frac{-2}{x^3} = \frac{2}{\ln 10}\cdot\frac{1}{x^3}$

$f^{(4)}(x) = f''''(x) = \left(\frac{2}{\ln 10}\cdot\frac{1}{x^3}\right)' = \frac{2}{\ln 10}\cdot\frac{-3}{x^4} = \frac{-6}{\ln 10\, x^4}$

$f^{(4)}(2) = \frac{-6}{\ln 10\cdot 2^4} = \frac{-6}{16\ln 10}$

Primjer 11. $f(x) = \cos^2(2x)$, $f''\left(\frac{\pi}{2}\right) = ?$

$f'(x) = (\cos^2(2x))' = ((\cos(2x))^2)' = \begin{cases} f = x^2 & f' = 2x \\ g = \cos(2x) \end{cases} = 2\cos(2x)\cdot(\cos(2x))'$

$f'(g(x))\cdot g'(x)$

$= 2\cos(2x)\cdot(-\sin(2x))\cdot 2 = -4\sin(2x)\cos(2x)$

$f''(x) = -4\left((\sin(2x))'\cdot\cos(2x) + \sin(2x)\cdot(\cos(2x))'\right)$

$= -4(\cos(2x)\cdot 2\cdot\cos(2x) + \sin(2x)\cdot(-\sin(2x))\cdot 2)$

$= -8(\cos^2(2x) - \sin^2(2x))$

$\Rightarrow f''\left(\frac{\pi}{2}\right) = -8(\cos^2(\pi) - \sin^2(\pi)) = -8((-1)^2 - 0^2) = -8$

Višestrukom primjenom L'Hopitalovog pravila odrediti sljedeće:

Primjer 12. $\lim_{n\to\infty} \frac{n^2}{2^n} = \left[\frac{\infty}{\infty}\right] \overset{L'H}{=} \lim_{n\to\infty} \frac{(n^2)'}{(2^n)'} = \lim_{n\to\infty} \frac{2n}{2^n \ln 2} = \left[\frac{\infty}{\infty}\right] =$

Vidi primjer 17. iz seminara 9!

$\overset{L'H}{=} \lim_{n\to\infty} \frac{(2n)'}{(2^n \ln 2)'} = \lim_{n\to\infty} \frac{2}{2^n \ln 2 \cdot \ln 2} = \frac{2}{\infty} = 0$

Primjer 13. $\lim_{x\to 1+} \ln x \ln(x-1) = [\ln 1_+ \cdot \ln 0_+] = [0_+ \cdot (-\infty)]$ NEODREĐENI OBLIK

ALI NAMA TREBA NEODREĐENI OBLIK $\frac{0}{0}, \frac{\infty}{\infty}$

$= \lim_{x\to 1_+} \frac{\ln(x-1)}{\frac{1}{\ln x}} = \left[\frac{-\infty}{+\infty}\right] \overset{L'H}{=} \lim_{x\to 1_+} \frac{\frac{1}{x-1}}{\frac{-1}{(\ln x)^2}\cdot\frac{1}{x}} = \lim_{x\to 1_+}\left(-\frac{x(\ln x)^2}{x-1}\right) = \left[-\frac{1\cdot 0^2}{0_+} = \frac{0}{0}\right]$

FORMALNO: $0_+ \cdot (-\infty) = \frac{1}{\frac{1}{0_+}} \cdot (-\infty) = \frac{1}{+\infty}\cdot(-\infty) = \frac{-\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to 1_+} - \frac{(\ln x)^2 + x \cdot 2\ln x \cdot \frac{1}{x}}{1} = \lim_{x\to 1_+} - \frac{(\ln x)^2 + 2\ln x}{1} = -\frac{0^2+0}{1} = 0$

Izračunati derivacije funkcija koje slijede:

Primjer 14. $f(x) = \arcsin(\ln(x^2-4))$, $f' = \left\{ \begin{array}{l} g(x) = \ln(x^2-4) \\ f(x) = \arcsin x, \; f'(x) = \frac{1}{\sqrt{1-x^2}} \end{array} \right\} = \underbrace{\frac{1}{\sqrt{1-(g(x))^2}}}_{f'(g(x))} \cdot g'(x) =$

$\underbrace{f(g(x))}$

$= \frac{1}{\sqrt{1-(\ln(x^2-4))^2}} \cdot (\ln(x^2-4))' = \frac{1}{\sqrt{1-(\ln(x^2-4))^2}} \cdot \frac{2x}{x^2-4}$

$\underbrace{(\ln(x^2-4))'}_{f(g(x))} = \left\{ \begin{array}{ll} g(x) = x^2-4 & g'(x) = 2x \\ f(x) = \ln x & f'(x) = \frac{1}{x} \end{array} \right\} = \underbrace{\frac{1}{x^2-4} \cdot 2x}_{f'(g(x))\cdot g'(x)}$

Primjer 15. $f(x) = \ln\ln(x^2+x)$

$[\ln(\ln(x^2+x))]' = \frac{1}{\ln(x^2+x)} \cdot [\ln(x^2+x)]' = \frac{1}{\ln(x^2+x)} \cdot \frac{1}{x^2+x} \cdot [x^2+x]' = \frac{1}{\ln(x^2+x)} \cdot \frac{1}{x^2+x} \cdot (2x+1)$

$f'(x) = \frac{1}{\ln(x^2+x)} \cdot \frac{2x+1}{x^2+x}$

Primjer 16. $f(x) = \frac{1}{\cos^2(3x)} = \frac{1}{(\cos(3x))^2} = (\cos(3x))^{-2}$

$$\begin{aligned} f'(x) &= -2(\cos(3x))^{-3}\cdot(\cos(3x))' \\ &= -2(\cos(3x))^{-3}\cdot(-\sin(3x))\cdot(3x)' \\ &= +2(\cos(3x))^{-3}\cdot\sin(3x)\cdot 3 \\ &= \frac{6\sin(3x)}{\cos^3(3x)} \end{aligned}$$

Primjer 17. $f(x) = x\,e^{\frac{1}{x^2-1}}$ *Vidi* `http://lavica.fesb.hr/mat1/vjezbe/node109.html`

$$f'(x) = \left(x\cdot e^{\frac{1}{x^2-1}}\right)' = x'\cdot e^{\frac{1}{x^2-1}} + x\cdot\left(e^{\frac{1}{x^2-1}}\right)' = 1\cdot e^{\frac{1}{x^2-1}} + x\cdot e^{\frac{1}{x^2-1}}\cdot\frac{-2x}{(x^2-1)^2} = (*)$$

$$\underbrace{\left(e^{\frac{1}{x^2-1}}\right)'}_{(f(g(x)))'} = \left\{\begin{matrix} g(x)=\frac{1}{x^2-1}, & g'(x) = -\frac{2x}{(x^2-1)^2} \\ f(x)=e^x, & f'(x)=e^x \end{matrix}\right\} = \boxed{e^{\frac{1}{x^2-1}}\cdot\frac{-2x}{(x^2-1)^2}}$$

$$f'(g(x))\cdot g'(x)$$

$$f'(x) = (*) = e^{\frac{1}{x^2-1}}\left(1-\frac{2x^2}{(x^2-1)^2}\right) = e^{\frac{1}{x^2-1}}\left(\frac{(x^2-1)^2-2x^2}{(x^2-1)^2}\right) = e^{\frac{1}{x^2-1}}\cdot\frac{x^4-4x^2+1}{(x^2-1)^2}$$

Primjer 18. $f(x) = (\cos x)^{\sin x}$

OBLIK $(f(x))^{g(x)}$ NEMAMO U TABLICI, NITI U PRAVILIMA

SVODIMO NA OBLIK $e^{\text{nešto}}$. EVO KAKO: $f(x) = e^{\ln f(x)}$

$f(x)^{g(x)} = \left(e^{\ln f(x)}\right)^{g(x)} = e^{g(x)\ln f(x)}$

$(e^a)^b = e^{a\cdot b}$

$$\begin{aligned} f(x) &= \left(e^{\ln(\cos x)}\right)^{\sin x} \\ &= e^{\sin x\ln(\cos x)} \end{aligned}$$

$$f'(x) = \left(e^{\sin x\ln(\cos x)}\right)' = \left\{\begin{matrix} g(x)=\sin x\ln(\cos x) & \\ f(x)=e^x, & f'(x)=e^x \end{matrix}\right\} = e^{\sin x\ln(\cos x)}\cdot(\sin x\ln(\cos x))' =$$

$$= \underbrace{e^{\sin x\ln(\cos x)}}_{\parallel}\cdot\left(\cos x\ln(\cos x) + \sin x\cdot\frac{1}{\cos x}\cdot(-\sin x)\right)$$

$$= (\cos x)^{\sin x}\left(\cos x\ln(\cos x) - \frac{\sin^2 x}{\cos x}\right)$$

Za samostalnu vježbu derivirati sljedeće funkcije:

(1) $f(x) = (x+2)^3$
(2) $f(x) = 2x + \sqrt{x^2+x}$
(3) $f(x) = \sqrt{4+x} - \sqrt{4-x}$
(4) $f(x) = \ln(\cos x)$
(5) $f(x) = \arcsin\frac{x-1}{x}$
(6) $f(x) = \sqrt{4x-x^2}\ln(x-2)$
(7) $f(x) = x^2\ln x$

Usporediti derivaciju funkcije $f_1(x) = \tan x$ sa $f_2(x) = \frac{\sin x}{\cos x}$

Primjenom L'Hopitalovog pravila riješiti:

(1) $\lim\limits_{x\to-\infty}\frac{x^3+3x-1}{2x^2-2x+2}$ (*derivirati dok je limes neodređenog oblika* $\frac{\infty}{\infty}$)
(2) $\lim\limits_{x\to0}\frac{\sin x}{x}$
(3) $\lim\limits_{x\to+\infty}\frac{x^3}{e^x}$
(4) $\lim\limits_{x\to0}\frac{\sin 3x}{\sqrt{x+2}-x-\sqrt{2}}$
(5) $\lim\limits_{x\to0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

Za vježbu pokušajte riješiti neke limese iz seminara 8 do 11 korištenjem L'Hopitalovog pravila. Naravno, rezultat treba biti isti.