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Integriranje kompozicije racionalne s trigonometrijskom funkcijom

Neka je $R(x, y)$ neka racionalna funkcija varijabli x i y. Tada sa $R(\sin x, \cos x)$ označavamo kompoziciju racionalne s trigonometrijskim funkcijama. Na primjer, ako je $R(x, y) = \dfrac{x+y}{y^2 - xy}$, tada je $R(\sin x, \cos x) = \dfrac{\sin x + \cos x}{\cos^2 x - \sin x \cos x}$. Dalje, ako je na primjer $R(\sin x, \cos x) = \tan x = \dfrac{\sin x}{\cos x}$ tada možemo uzimati $R(x, y) = \dfrac{x}{y}$.

Neke kompozicije racionalne i trigonometrijske funkcije lagano je integrirati: bilo tablično ili tehnikama izloženim ranije. Na primjer, ranije smo izračunali $\int \sin^2 x \, \mathrm{d}x = \frac{x}{2} - \frac{1}{4}\sin(2x)$, $\int \mathrm{d}x / \cos^2 x = \tan x + C$, ... Sada ćemo dati proceduru za integriranje složenijih kompozicija racionalne i trigonometrijske funkcije. Ovu vrstu integrala riješavamo jednom od predloženih supstitucija:

Supstitucija	x	$\sin x$	$\cos x$	$\tan x$	$\cot x$	Kada koristimo?
$t = \tan \frac{x}{2}$	$x = 2 \arctan t$	$\dfrac{2t}{1+t^2}$	$\dfrac{1-t^2}{1+t^2}$	$\dfrac{2t}{1-t^2}$	$\dfrac{1-t^2}{2t}$	U najopćenitijem slučaju.
$t = \tan x$	$x = \arctan t$	$\pm\sqrt{\dfrac{t^2}{1+t^2}}$	$\pm\sqrt{\dfrac{1}{1+t^2}}$	t	$\dfrac{1}{t}$	Ako vrijedi : $R(-\cos x, -\sin x) = R(\cos x, \sin x)$.
$t = \sin x$	$x = \arcsin t$	t	$1 - t^2$			U slučaju $\int R(\sin x) \cos x \, \mathrm{d}x$.
$t = \cos x$	$x = \arccos t$	$1 - t^2$	t			U slučaju $\int R(\cos x) \sin x \, \mathrm{d}x$.

Primjer.

$$\int \frac{\sin x + \cos x}{\cos^2 x - \sin x \cos x} \mathrm{d}x = \left\{ \begin{array}{rr} t = \tan \frac{x}{2}, & x = 2 \arctan t \\ & \mathrm{d}x = \dfrac{2\mathrm{d}t}{1+t^2} \end{array} \right\} = \int \frac{\dfrac{2t}{1+t^2} + \dfrac{1-t^2}{1+t^2}}{\left(\dfrac{1-t^2}{1+t^2}\right)^2 - \dfrac{2t}{1+t^2} \cdot \dfrac{1-t^2}{1+t^2}} \cdot \frac{2\mathrm{d}t}{1+t^2} = \ldots$$

$$-t^2 - 2t + 1 = 0$$
$$t_{1,2} = \frac{2 \pm \sqrt{4+4}}{-2} = \frac{2 \pm 2\sqrt{2}}{-2}$$
$$t_1 = -1-\sqrt{2}, \quad t_2 = -1+\sqrt{2}$$

$$= \int \frac{\dfrac{1+2t-t^2}{\cancel{1+t^2}} \cdot 2}{\dfrac{1-2t^2+t^4-2t+2t^3}{\cancel{(1+t^2)^2}} \cdot \cancel{(1+t^2)}} dt = 2 \int \frac{1+2t-t^2}{(1-t^2)^2 - 2t(1-t^2)} dt = 2 \int \frac{1+2t-t^2}{(1-t^2)(1-t^2-2t)} dt$$

$$\left. \begin{array}{l} \dfrac{1+2t-t^2}{(1-t^2)(1-t^2-2t)} = \dfrac{A}{1-t} + \dfrac{B}{1+t} + \dfrac{C}{t+1+\sqrt{2}} + \dfrac{D}{t+1-\sqrt{2}} \\ = \dfrac{-\frac{1}{2}}{1-t} + \dfrac{-\frac{1}{2}}{1+t} + \dfrac{\frac{1}{\sqrt{2}}}{t+1+\sqrt{2}} + \dfrac{-\frac{1}{\sqrt{2}}}{t+1-\sqrt{2}} \end{array} \right\} \Rightarrow$$

$$= 2\left[-\frac{1}{2} \int \frac{1}{1-t} dt - \frac{1}{2} \int \frac{1}{1+t} dt + \frac{1}{\sqrt{2}} \int \frac{dt}{t+1+\sqrt{2}} - \frac{1}{\sqrt{2}} \int \frac{dt}{t+1-\sqrt{2}} \right]$$

$$= \ln|1-t| - \ln|t+1| + \frac{1}{\sqrt{2}} \ln|t+1+\sqrt{2}| - \frac{1}{\sqrt{2}} \ln|t+1-\sqrt{2}| + C$$

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$$\int \frac{\cos x}{(1+\cos x)^3} \mathrm{d}x = \left\{ \begin{array}{l} t = \tan \frac{x}{2}, \ x = \arctan t \\ dx = \dfrac{2dt}{1+t^2} \end{array} \right\} = \int \frac{\dfrac{1-t^2}{1+t^2}}{\left(1 + \dfrac{1-t^2}{1+t^2}\right)^3} \cdot \frac{2dt}{1+t^2} = 2 \int \frac{\dfrac{1-t^2}{\cancel{1+t^2}}}{\dfrac{(1+t^2+1-t^2)^3}{(1+t^2)^{\cancel{3}}}} \cdot \frac{dt}{\cancel{1+t^2}} = 2 \int \frac{(1-t^2)(1+t^2)}{2^3} dt$$

$$= \frac{2}{8} \int (1-t^4) dt = \frac{1}{4}\left(t - \frac{t^5}{5}\right) = \frac{1}{4}\left(\tan\frac{x}{2} - \frac{1}{2}\left(\tan\frac{x}{2}\right)^5\right) + C$$

$$\int \frac{1+2t-t^2}{(1-t^2)(1-t^2-2t)}\,dt$$

INTEGRAL $\int \frac{\sin x+\cos x}{\cos^2 x-\sin x\cos x}\,dx = \left\{ t=\tan\frac{x}{2} \right\} = \ldots = 2\int \frac{1+2t-t^2}{(1-t^2)(1-t^2-2t)}\,dt$

$1-t^2=0$ za $t_{1,2}=\frac{0\pm\sqrt{0+4}}{2}=\pm 1$

$\Rightarrow 1-t^2=-(t^2-1)=-(t-1)(t+1)$

$1-t^2-2t=0$ za $t_{1,2}=\frac{2\pm\sqrt{4+4}}{-2}=-1\mp\sqrt{2},\,-1+\sqrt{2}$

$\Rightarrow (1-t^2-2t)=-(t^2+2t-1)=-(t+1-\sqrt{2})(t+1+\sqrt{2})$

RASTAV NA PARCIJALNE RAZLOMKE:

$$\Rightarrow \frac{1+2t-t^2}{(1-t^2)(1-t^2-2t)}=\frac{1+2t-t^2}{-(t-1)(t+1)\cdot(-1)(t+1-\sqrt{2})(t+1+\sqrt{2})}$$

$$=\frac{1+2t-t^2}{(t-1)(t+1)(t+1-\sqrt{2})(t+1+\sqrt{2})}$$

$$=\frac{A}{t-1}+\frac{B}{t+1}+\frac{C}{t+1-\sqrt{2}}+\frac{D}{t+1+\sqrt{2}}$$

JEDNAKOST TREBA VRIJEDITI ZA SVAKI t!
UVRŠTAVAMO ZA t NULTOČKE NAZIVNIKA:

Za $t=1$

$$1+2\cdot 1-1^2=A(1+1)(1+1-\sqrt{2})(1+1+\sqrt{2}) \Rightarrow A=\frac{2}{2(2-\sqrt{2})(2+\sqrt{2})}=\frac{2}{2\cdot(4-2)}=\frac{2}{4}=\frac{1}{2}$$

Za $t=-1$

$$+1+2\cdot(-1)-(-1)^2=B(-1-1)(-1+1-\sqrt{2})(-1+1+\sqrt{2}) \Rightarrow B=\frac{-2}{(-2)(-\sqrt{2})(\sqrt{2})}=\frac{-2}{-2\cdot(-2)}=\frac{-2}{4}=-\frac{1}{2}$$

Za $t=-1+\sqrt{2}$

$$1+2(-1+\sqrt{2})-(-1+\sqrt{2})^2=C(-1+\sqrt{2}-1)(-1+\sqrt{2}+1)(-1+\sqrt{2}+1+\sqrt{2})$$

$$1-2+2\sqrt{2}-1+2\sqrt{2}-2=C(-2+\sqrt{2})(\sqrt{2})\cdot 2\sqrt{2} \Rightarrow C=\frac{-4+4\sqrt{2}}{-8+4\sqrt{2}}\cdot\frac{8+4\sqrt{2}}{8+4\sqrt{2}}=\frac{-32+32-16\sqrt{2}+32\sqrt{2}}{-64+32}$$

$$C=\frac{16\sqrt{2}}{-32}=-\frac{\sqrt{2}}{2}$$

Za $t=-1-\sqrt{2}$

$$1+2(-1-\sqrt{2})-(-1-\sqrt{2})^2=D(-1-\sqrt{2}-1)(-1-\sqrt{2}+1)(-1-\sqrt{2}+1-\sqrt{2})$$

$$1-2-2\sqrt{2}-1-2\sqrt{2}-2=D(-2-\sqrt{2})(-\sqrt{2})(-2\sqrt{2}) \Rightarrow D=\frac{-4-4\sqrt{2}}{-8-4\sqrt{2}}\cdot\frac{8-4\sqrt{2}}{8-4\sqrt{2}}=\frac{-32+32+16\sqrt{2}-32\sqrt{2}}{-64+32}$$

$$D=\frac{-16\sqrt{2}}{-32}=\frac{\sqrt{2}}{2}$$

DAKLE:

$$\frac{1+2t-t^2}{(1-t^2)(1-t^2-2t)}=\frac{\frac{1}{2}}{t-1}+\frac{-\frac{1}{2}}{t+1}+\frac{-\frac{\sqrt{2}}{2}}{t+1-\sqrt{2}}+\frac{\frac{\sqrt{2}}{2}}{t+1+\sqrt{2}}$$

$$\int\frac{1+2t-t^2}{(1-t^2)(1-t^2-2t)}\,dt=\frac{1}{2}\int\frac{dt}{t-1}-\frac{1}{2}\int\frac{dt}{t+1}-\frac{\sqrt{2}}{2}\int\frac{dt}{t+1-\sqrt{2}}+\frac{\sqrt{2}}{2}\int\frac{dt}{t+1+\sqrt{2}}$$

$$=\frac{1}{2}\ln|t-1|-\frac{1}{2}\ln|t+1|-\frac{\sqrt{2}}{2}\ln|t+1-\sqrt{2}|+\frac{\sqrt{2}}{2}\ln|t+1+\sqrt{2}|+C$$

$$\Rightarrow \int\frac{\sin x+\cos x}{\cos^2 x-\sin x\cos x}\,dx=\left\{t=\tan\frac{x}{2}\right\}=2\int\frac{1+2t-t^2}{(1-t^2)(1-t^2-2t)}=\ln\left|\tan\frac{x}{2}-1\right|-\ln\left|\tan\frac{x}{2}+1\right|-\sqrt{2}\ln\left|\tan\frac{x}{2}+1-\sqrt{2}\right|$$
$$+\sqrt{2}\ln\left|\tan\frac{x}{2}+1+\sqrt{2}\right|+C$$

Ponekad se integrali koji uključuju trigonometrijske funkcije značajno pojednostavne kada se iskoristi pogodan trigonometrijski identitet. Među najkorisnijim identitetima su:

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$	$\cos^2 x = \frac{1}{2}(1+\cos 2x)$
$\sin^2 x + \cos^2 x = 1$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$

Primjer. Riješiti integral $\int \sin^5 x \, dx = \left\{ \sin^2 x = 1 - \cos^2 x \right\} = \int \underbrace{(1-\cos^2 x)}_{=\sin^2 x}\underbrace{(1-\cos^2 x)}_{=\sin^2 x} \sin x \, dx = \left\{ \begin{array}{ll} t = \cos x & x = \arccos t \\ dt = -\sin x \, dx & \end{array} \right\}$

$= \int (1-t^2)(1-t^2)(-dt) = -\int (1-2t^2+t^4)dt = -\left(t - \frac{2}{3}t^3 + \frac{t^5}{5}\right) = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

Riješiti integral $\int \cos^4 x \, dx = \int \underbrace{\frac{1}{2}(1+\cos 2x)}_{\cos^2 x} \cdot \underbrace{\frac{1}{2}(1+\cos 2x)}_{\cos^2 x} dx = \frac{1}{4}\int \left[1 + 2\cos(2x) + \cos^2(2x)\right] dx =$

$= \frac{1}{4}\underbrace{\int dx}_{=x} + \frac{1}{2}\underbrace{\int \cos(2x) dx}_{=\frac{1}{2}\sin(2x)} + \frac{1}{4}\int \cos^2(2x) dx = \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C$

$\int \cos^2(2x) dx = \int \frac{1}{2}(1+\cos(4x)) dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos(4x) dx$
$= \frac{1}{2}x + \frac{1}{2}\frac{\sin(4x)}{4}$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\cos^2(2x) = \frac{1}{2}(1+\cos 2\cdot(2x)) = \frac{1}{2}(1+\cos(4x))$

Samostalno za vježbu:

1. $\int \frac{\sin x \, dx}{\cos x + 5}$, 2. $\int \cos^5 x \, dx$, 3. $\int \sin^2 x \cos^2 x \, dx$, 4. $\int \frac{dx}{\sin x \,(2\cos^2 x - 1)}$, 5. $\int \frac{dx}{2\sin x - \cos x + 5}$