Integrali koji uključuju korijen

Integrali funkcija tipa $R(x^{\frac{m_1}{n_1}}, x^{\frac{m_2}{n_2}}, \ldots, x^{\frac{m_k}{n_k}})$ i $R(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_2}{n_2}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_k}{n_k}})$

Riješavamo tako da odredimo n, a to je zajednički višekratnik od n_1, n_2, do n_k. Zatim izvršimo supstituciju $x = t^n$.

Primjer. $\int \frac{\sqrt[2]{x}}{4+\sqrt[3]{x}}\mathrm{d}x = \left\{ \begin{array}{l} x = t^6, \quad t = \sqrt[6]{x} \\ dx = 6t^5 dt \end{array} \right\} = \int \frac{\sqrt[2]{t^6}}{4+\sqrt[3]{t^6}} \cdot 6t^5 dt = 6\int \frac{t^3 \cdot t^5}{4+t^2} dt = 6\int \frac{t^8}{t^2+4} dt = (*)$

JAVLJA SE $x^{\frac{1}{2}}$ i $x^{\frac{1}{3}}$

$\Downarrow \qquad \Downarrow$

$n_1 = 2 \qquad n_2 = 3$

NZV(2,3) = 6

$t^8 : (t^2+4) = t^6 - 4t^4 + 16t^2 - 64 + \frac{256}{t^2+4}$

$-(t^8+4t^6)$

$-4t^6$

$-(-4t^6-16t^4)$

$16t^4$

$-(16t^4+64t^2)$

$-64t^2$

$-(-64t^2-256)$

256

$$\Rightarrow (*) = 6\int \left(t^6 - 4t^4 + 16t^2 - 64 + \frac{256}{t^2+4}\right) dt = 6\left(\frac{t^7}{7} - 4\frac{t^5}{5} + 16\frac{t^3}{3} - 64t + 256 \cdot \frac{1}{2}\arctan\frac{t}{2}\right)$$

$$= \frac{6}{7}x^{\frac{7}{6}} - \frac{4}{5}x$$

Primjer. $\int \frac{\mathrm{d}x}{\sqrt[3]{x+1} + \sqrt[3]{(x+1)^2}} = \left\{ \begin{array}{l} t = x+1 \\ dt = dx \end{array} \right\} = \int \frac{dt}{t^{\frac{1}{3}} + t^{\frac{2}{3}}} = \left\{ \begin{array}{l} t = u^3 \\ dt = 3u^2 du \end{array} \right\} = \int \frac{3u^2 du}{u+u^2} = 3\int du - 3\int \frac{u\,du}{u^2+u} =$

$3u^2 : (u^2+u) = 3 - \frac{3u}{u^2+u}$

$-(3u^2+3u)$

$-3u$

$$= 3u - 3\int \frac{du}{u+1} = 3u - 3\ln|u+1| =$$

$$= 3\sqrt[3]{t} - 3\ln|\sqrt[3]{t}+1| =$$

$$= 3\sqrt[3]{x+1} - 3\ln|\sqrt[3]{x+1}+1| + C$$

$\frac{u}{u^2+u} = \frac{A}{u} + \frac{B}{u+1} = \frac{1}{u+1}$

$= \frac{Au + A + Bu}{u(u+1)}$

$\Rightarrow \left\{ \begin{array}{l} u = Au + Bu \Rightarrow B = 1 \\ 0 = A \end{array} \right.$

Integrali funkcija tipa $\dfrac{polinom(x)}{\sqrt{ax^2+bx+c}}$ i $R\left(x,\ \sqrt{ax^2+bx+c}\right)$

Obratite pažnju na 3 tablična integrala koja nam koriste kod riješavanja integrala ovog tipa:

- $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C = -\arccos x + C$
- $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \cosh^{-1} x + C = \ln\left|x+\sqrt{x^2-1}\right| + C$
- $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \sinh^{-1} x + C = \ln\left|x+\sqrt{x^2+1}\right| + C$

Funkcija $\cosh^{-1}$ je area cosinus hiperbolni koja se negdje označaava sa arch. Funkcija $\sinh^{-1}$ je area sinus hiperbolni koja se negdje označaava sa arsh.

Svaki integral oblika $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}}$ može se svesti na jedan od gornja tri tako da prvo transformiramo izraz $ax^2+bx+c = \pm\left(\text{član sa } x\right)^2 \pm broj$, a zatim izlučimo $broj$ tako da $član\, bez\, x$ bude 1.

Primjer.

$$
\begin{aligned}
4x^2+6x+3 &= (2x)^2 + \underbrace{2\cdot 2x\cdot \underline{?}}_{=6x} + 3 \\
&= (2x)^2 + \underbrace{2\cdot 2x\cdot \frac{3}{2}}_{=6x} + \underbrace{\left(\frac{3}{2}\right)^2 + \underline{?}}_{=3} \\
&= \left[(2x)^2 + 2\cdot 2x\cdot\frac{3}{2} + \left(\frac{3}{2}\right)^2\right] + \left(3-\frac{9}{4}\right) \\
&= \left(2x+\frac{3}{2}\right)^2 + \frac{3}{4} \\
&= \frac{3}{4}\left[\left(\frac{2x+\frac{3}{2}}{\frac{\sqrt{3}}{2}}\right)^2 + 1\right]
\end{aligned}
$$

Zadatak. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{4x^2+6x+3}} = \int \frac{dx}{\sqrt{\left(2x+\frac{3}{2}\right)^2 + \frac{3}{4}}} = \left\{ \begin{array}{l} t = 2x+\frac{3}{2} \\ dt = 2\,dx,\ dx = \frac{dt}{2} \end{array} \right\} =$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t^2 + \left(\sqrt{\frac{3}{4}}\right)^2}} = \frac{1}{2}\ln\left|t+\sqrt{t^2+\frac{3}{4}}\right| + c =$$

$$= \frac{1}{2}\ln\left|2x+\frac{3}{2}+\sqrt{\left(2x+\frac{3}{2}\right)^2+\frac{3}{4}}\right| + C$$

Transformiraj izraz tako da bude pogodan za integriranje pod korijenom, kao u prethodnom primjeru:

$$
\begin{aligned}
4x^2 + 4x - 3 &= (2x)^2 + \underbrace{2 \cdot 2x \cdot ?}_{=4x} - 3 \\
&= (2x)^2 + \underbrace{2 \cdot 2x \cdot 1}_{=4x} + \underbrace{1^2 - 1^2 - 3}_{=-3} \\
&= (2x+1)^2 - 4 \\
&= \\
&= 4\left[\left(\frac{2x+1}{2}\right)^2 - 1\right]
\end{aligned}
$$

Zadatak.

$$\int \frac{dx}{\sqrt{4x^2+4x-3}} = \int \frac{dx}{\sqrt{(2x+1)^2-4}} = \left\{ \begin{array}{l} t = 2x+1 \\ dt = 2dx \\ dx = \frac{dt}{2} \end{array} \right\} = \frac{1}{2}\int \frac{dt}{\sqrt{t^2-4}} = \frac{1}{2}\ln\left|t + \sqrt{t^2-4}\right|$$

$$= \frac{1}{2}\ln\left|2x+1+\sqrt{(2x+1)^2-4}\right| + C$$

$$= \frac{1}{2}\ln\left|2x+1+\sqrt{4x^2+4x-3}\right| + C$$

Integral funkcije tipa $\dfrac{polinom(x)}{\sqrt{ax^2+bx+c}}$, gdje je $polinom(x)$ stupnja n riješavamo tako da postavimo jednadžbu

$$\int \frac{polinom(x)}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x = \left(A_0 + A_1 x + \ldots + A_{n-1}x^{n-1}\right)\sqrt{ax^2+bx+c} + B\int \frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}},$$

a zatim nepoznanice A_0, A_1, ..., A_{n-1} i B odredimo iz derivacije gornje jednadžbe.

Primjer. $\displaystyle\int \frac{x^2\,\mathrm{d}x}{\sqrt{9-4x^2}} = ?$

Polinom u brojniku je stupnja $n = 2$. Postavimo jednadžbu:

$$\int \frac{x^2\,\mathrm{d}x}{\sqrt{9-4x^2}} = (A_0 + A_1 x)\sqrt{9-4x^2} + B\int \frac{\mathrm{d}x}{\sqrt{9-4x^2}}$$

Derivacija prethodne jednadžbe:

$$\begin{aligned}\frac{x^2}{\sqrt{9-4x^2}} &= A_1\sqrt{9-4x^2} + (A_0 + A_1 x)\cdot\frac{1}{2}\cdot\frac{-4\cdot 2x}{\sqrt{9-4x^2}} + \frac{B}{\sqrt{9-4x^2}}\\ &= \frac{A_1\left(9-4x^2\right) - 4x\left(A_0 + A_1 x\right) + B}{\sqrt{9-4x^2}}\end{aligned}$$

Nepoznanice ćemo odrediti tako da usporedimo članove s istom potencijom od x:

$$\begin{aligned} x^2 &: \quad 1 = -4A_1 - 4A_1\\ x &: \quad 0 = -4A_0\\ 1 &: \quad 0 = 9A_1 + B\end{aligned}$$

Dakle,

$$\int \frac{x^2\,\mathrm{d}x}{\sqrt{9-4x^2}} = -\frac{1}{8}x\sqrt{9-4x^2} + \frac{9}{8}\int \frac{\mathrm{d}x}{\sqrt{9-4x^2}}$$

i preostaje odrediti još samo $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{9-4x^2}}$ ...

Prvo napravimo prikladnu transformaciju izraza

$$9 - 4x^2 = 9\left[1 - \left(\frac{2x}{3}\right)^2\right]$$

$$\int \frac{\mathrm{d}x}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{9\left(1-\left(\frac{2x}{3}\right)^2\right)}} = \frac{1}{3}\int \frac{dx}{\sqrt{1-\left(\frac{2x}{3}\right)^2}} = \left\{\begin{array}{l} t = \frac{2x}{3}\\ dt = \frac{2}{3}dx,\ dx = \frac{3}{2}dt\end{array}\right\} =$$

$$= \frac{1}{3}\cdot\frac{3}{2}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{2}\arcsin t + C = \frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + C$$

$$\Longrightarrow \quad \int \frac{x^2\,\mathrm{d}x}{\sqrt{9-4x^2}} = -\frac{1}{8}x\sqrt{9-4x^2} + \frac{9}{16}\arcsin\left(\frac{2x}{3}\right) + C$$

Binomni integral

Integral oblika

$$\int x^m (a+bx^n)^p \,\mathrm{d}x, \quad m,n,p\in\mathbb{Q},\ a,b\in\mathbb{R}$$

riješavamo prema sljedećim slučajevima:

$x^m(a+bx^n)^p$	Test	Supstitucija
1. slučaj	$p\in\mathbb{Z}$	$x=t^k$, gdje je k zajednički nazivnik od m i n
2. slučaj	$\frac{m+1}{n}\in\mathbb{Z}$	$a+bx^n=t^k$, gdje je k nazivnik od p
3. slučaj	$\frac{m+1}{n}+p\in\mathbb{Z}$	$ax^{-n}+b=t^k$, gdje je k nazivnik od p

Riješiti sljedeće integrale:

1. $\int \frac{\mathrm{d}x}{x^4\sqrt{1+x^2}}$
2. $\int \frac{x^3\,\mathrm{d}x}{(9-x^2)\sqrt{9-x^2}}$
3. $\int x^5\sqrt[7]{2-3x^6}\,\mathrm{d}x$
4. $\int \frac{\mathrm{d}x}{\sqrt{x}\,(\sqrt[4]{x}+1)^{10}}$
5. $\int x^{-11}(1+x^4)^{-\frac{1}{2}}\,\mathrm{d}x$

(1) $\int x^{-4}(1+x^2)^{-\frac{1}{2}}\,dx = (*)$ (m = −4, n = 2, p = −1/2)

1. slučaj $-\frac{1}{2}\in\mathbb{Z}$ NE

2. slučaj $\frac{-4+1}{2}\in\mathbb{Z}$ NE

$\Leftarrow$ 3. slučaj $\frac{-4+1}{2}-\frac{1}{2}\in\mathbb{Z}$ DA

SUPSTITUCIJA:

$1\cdot x^{-2}+1=t^2$

$x^{-2}+1=t^2 \longrightarrow t=\sqrt{1+x^{-2}}=\sqrt{1+x^{-2}}=(1+x^{-2})^{\frac{1}{2}}$

$x^{-2}=t^2-1$

$x^2=\frac{1}{t^2-1}$

$x=\sqrt{\frac{1}{t^2-1}}$, $x=(t^2-1)^{-\frac{1}{2}}$, $dx=-\frac{1}{2}(t^2-1)^{-\frac{3}{2}}\cdot 2t\,dt$

$$(*)=\int (t^2-1)^2\left(1+\frac{1}{t^2-1}\right)^{-\frac{1}{2}}\cdot(-1)(t^2-1)^{-\frac{3}{2}}\cdot t\,dt=-\int (t^2-1)^{\frac{1}{2}}\left(\frac{t^2-1+1}{t^2-1}\right)^{-\frac{1}{2}}t\,dt=$$

$$=-\int (t^2-1)^{\frac{1}{2}}\left(\frac{t^2}{t^2-1}\right)^{-\frac{1}{2}}t\,dt=-\int (t^2-1)^{\frac{1}{2}}(t^2-1)^{\frac{1}{2}}(t^2)^{-\frac{1}{2}}t\,dt=$$

$$=-\int (t^2-1)^{\frac{1}{2}+\frac{1}{2}}\,t^{2\cdot(-\frac{1}{2})+1}\,dt=-\int t^2-1\,dt$$

$$=-\frac{t^3}{3}+t=-\frac{(1+x^{-2})^{\frac{3}{2}}}{3}+\sqrt{1+x^{-2}}+C$$

(2) $\int \frac{x^3 dx}{(9-x^2)\sqrt{9-x^2}} = \int x^{3}(9-x^{2})^{-\frac{3}{2}} dx$ (m = 3, n = 2, p = $-\frac{3}{2}$)

1. slučaj $-\frac{3}{2} \in \mathbb{Z}$ NE

2. slučaj $\frac{3+1}{2} \in \mathbb{Z}$ DA

⇓

SUPSTITUCIJA:

$9 - x^2 = t^2$

$x^2 = 9 - t^2$

$x = \sqrt{9-t^2} = (9-t^2)^{\frac{1}{2}}$

$dx = \frac{1}{2}(9-t^2)^{-\frac{1}{2}}(-2t)\,dt$

$t = \sqrt{9-x^2}$

$= -\int (9-t^2)^{\frac{1}{2}\cdot 3}\, t^{2\cdot(-\frac{3}{2})}(9-t^2)^{-\frac{1}{2}} \cdot t\,dt =$

$= -\int (9-t^2)^{\frac{3}{2}-\frac{1}{2}}\, t^{-3} \cdot t^1\, dt = -\int (9-t^2)^1 \cdot t^{-2}\, dt$

$= -\int \frac{9-t^2}{t^2}\, dt = -\int \frac{9}{t^2} - \frac{t^2}{t^2}\, dt = -9\int \frac{1}{t^2}dt + \int dt =$

$= -9\, \frac{t^{-1}}{-1} + t = \frac{9}{t} + t = \frac{9}{\sqrt{9-x^2}} + \sqrt{9-x^2} + C$

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(3) $\int x^5 \sqrt[7]{2-3x^6}\, dx = \int x^{5}(2-3x^{6})^{\frac{1}{7}} dx$ (m = 5, n = 6, p = $\frac{1}{7}$; $2-3x^6 = t^7$)

1. slučaj $\frac{1}{7} \in \mathbb{Z}$ NE

2. slučaj $\frac{5+1}{6} \in \mathbb{Z}$ DA

SUPSTITUCIJA:

$2 - 3x^6 = t^7 \;/'$

$-3 \cdot 6 x^5 dx = 7t^6 dt$

$x^5 dx = -\frac{7}{18} t^6 dt$

$t = (2-3x^6)^{\frac{1}{7}}$

$= -\frac{7}{18}\int (t^7)^{\frac{1}{7}} \cdot t^6 dt = -\frac{7}{18}\int t \cdot t^6 dt$

$= -\frac{7}{18}\int t^7 dt = -\frac{7}{18}\,\frac{t^8}{8} + C = -\frac{7}{18}\cdot\frac{1}{8}(2-3x^6)^{\frac{8}{7}} + C$

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(4) $\int \frac{dx}{\sqrt{x}\,(\sqrt[4]{x}+1)^{10}} = \int x^{-\frac{1}{2}}(1+x^{\frac{1}{4}})^{-10} dx$ (m, n, p)

1. slučaj $p \in \mathbb{Z}$ DA

SUPSTITUCIJA:

$x = t^4 \longrightarrow t = \sqrt[4]{x}$

$dx = 4t^3 dt$

$= 4\int (t^4)^{-\frac{1}{2}}\left(1+(t^4)^{\frac{1}{4}}\right)^{-10} t^3 dt =$

$= 4\int t^{-2}(1+t)^{-10} t^3 dt = 4\int t^{-2+3}(1+t)^{-10} dt =$

$= 4\int t(1+t)^{-10} dt = 4\int \frac{t}{(1+t)^{10}} dt = \left\{ \begin{array}{l} 1+t = u \\ dt = du \\ t = u-1 \end{array} \right\} = 4\int \frac{u-1}{u^{10}} du = 4\int \frac{u}{u^{10}} - \frac{1}{u^{10}} du =$

$= 4\int u^{-9} du - 4\int u^{-10} du = 4\cdot\frac{u^{-8}}{-8} - 4\,\frac{u^{-9}}{-9} = -\frac{1}{2}\,\frac{1}{u^8} + \frac{4}{9}\,\frac{1}{u^9} =$

$= -\frac{1}{2}\,\frac{1}{(t+1)^8} + \frac{4}{9}\,\frac{1}{(t+1)^9} = -\frac{1}{2}\,\frac{1}{(\sqrt[4]{x}+1)^8} + \frac{4}{9}\,\frac{1}{(\sqrt[4]{x}+1)^9} + C$

(5) $$\int x^{-11}\left(1+x^{4}\right)^{-\frac{1}{2}}dx =$$

($m = -11$, $n = 4$, $p = -\frac{1}{2}$)

1° $-\frac{1}{2} \in \mathbb{Z}$ NE

2° $\frac{-11+1}{4} \in \mathbb{Z}$ NE

3° $\frac{-11+1}{4} - \frac{1}{2} \in \mathbb{Z}$ DA

SUPSTITUCIJA:

$$x^{-4}+1 = t^{2} \rightarrow t = \left(1+x^{-4}\right)^{\frac{1}{2}}$$

$$x^{-4} = t^{2}-1$$

$$x = \left(t^{2}-1\right)^{-\frac{1}{4}}$$

$$dx = -\frac{1}{4_{2}}\left(t^{2}-1\right)^{-\frac{5}{4}} \cdot 2t\,dt$$

$$= -\frac{1}{2}\int\left(t^{2}-1\right)^{-\frac{1}{4}\cdot(-11)}\left(1+\frac{1}{t^{2}-1}\right)^{-\frac{1}{2}}\left(t^{2}-1\right)^{-\frac{5}{4}}t\,dt$$

$$= -\frac{1}{2}\int\left(t^{2}-1\right)^{\frac{11}{4}}\left(\frac{t^{2}-1+1}{t^{2}-1}\right)^{-\frac{1}{2}}\left(t^{2}-1\right)^{-\frac{5}{4}}t\,dt$$

$$= -\frac{1}{2}\int\left(t^{2}-1\right)^{\frac{11}{4}-\frac{5}{4}}\frac{t^{2\cdot -\frac{1}{2}}}{\left(t^{2}-1\right)^{-\frac{1}{2}}}\cdot t\,dt$$

$$= -\frac{1}{2}\int\left(t^{2}-1\right)^{\frac{6}{4}}\left(t^{2}-1\right)^{\frac{1}{2}}\cancel{t^{-1}}\cdot\cancel{t}\,dt$$

$$= -\frac{1}{2}\int\left(t^{2}-1\right)^{\frac{6}{4}+\frac{1}{2}}dt = -\frac{1}{2}\int\left(t^{2}-1\right)^{\frac{8}{4}}dt = -\frac{1}{2}\int\left(t^{2}-1\right)^{2}dt =$$

$$= -\frac{1}{2}\int t^{4}-2t^{2}+1\,dt = -\frac{1}{2}\left[\frac{t^{5}}{5}-2\cdot\frac{t^{3}}{3}+t\right]+C$$

$$= -\frac{1}{2}\left[\frac{\left(1+x^{-4}\right)^{\frac{5}{2}}}{5}-\frac{2}{3}\cdot\left(1+x^{-4}\right)^{\frac{3}{2}}+\sqrt{1+x^{-4}}\right]+C$$

Nekoliko dodatnih primjera primjene integrala

Primjer iz fizike (jednoliko ubrzano gibanje).

Poznata je jednažba koja vezuje brzinu v i akceleraciju a kod jednoliko ubrzanog gibanja: $v(t) = a\,t$, gdje je t vrijeme proteklo od trenutka ubrzanja s početne brzine $v(0) = 0$. Također, poznato je da se put s koji je prijeđen pravocrtnim gibanjem može izraziti u ovisnosti od brzine $v(t)$ kao integral: $s(t) = \int_0^t v(\tau)\,\mathrm{d}\tau$. Potrebno je izraziti put s kao funkciju vremena pri jednoliko ubrzanom gibanju.

$$s(t) = \int_{t_0}^{t} v(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} a\,\tau\,\mathrm{d}\tau = a\int_{t_0}^{t} \tau\,d\tau = a\left(\frac{\tau^2}{2}\right)\Big|_{t_0}^{t} = a\left(\frac{t^2}{2} - \frac{t_0^2}{2}\right)$$

AKO JE $t_0 = 0$

$$= a\frac{t^2}{2}$$

Primjer iz fizike (opruga).

Potrebno je izračunati rad za natezanje opruge iz neopterećenog položaja $s = 0$ do položaja $s = x$. Poznato je da opruga istegnuta za udaljenost x od neopterećenog položaja djeluje silom u smjeru suprotnom istegnuću sa iznosom $F = k\,x$, gdje je k konstanta elastičnosti opruge. Također je poznato da se rad obavljen na putu s pod djelovanjem sile F računa kao $W = \int F\,\mathrm{d}s$. Potrebno je izraziti W kao funkciju udaljenosti od položaja ravnoteže x.

$$W = \int_0^x F\,\mathrm{d}s = \int_0^x k\,s\,\mathrm{d}s = k\int_0^x s\,ds = k\left(\frac{s^2}{2}\right)\Big|_0^x = \frac{kx^2}{2}$$

Duljina luka krivulje

Vidi knjiga Napomenu 4.3.6.b) na str. 227.

Teorem 4.3.7. krivulja opisana koordinatama točaka $r(t)=(\phi(t),\psi(t))$

koordinate ovise o vremenu t

derivacije $\phi'(t)$ i $\psi'(t)$ su komponente brzine

veličina $\sqrt{(\phi'(t))^2+(\psi'(t))^2}$ je brzina kao broj

$$\boxed{\begin{array}{l}\text{prevaljeni put}\\ \text{od vremena } t_1\\ \text{do } \text{ -''- } \ t_2\end{array}} = \int_{t_1}^{t_2} \sqrt{(\phi'(t))^2+(\psi'(t))^2}\, dt$$