Dalje između ostalog donosimo nekoliko čestih skalarnih funkcija u ravnini čiji prikaz u 3 dimenzije određuje popularne geometrijske oblike. Nakon što obradimo odgovarajuće gradivo na svakom od sljedećih 5 primjera provedite postupak traženja lokalnih ekstrema.

Ravnina: $z = Ax + By + C$

Ispitati funkciju $f(x, y) = x + 2y + 1$.

$\mathcal{D}(f) = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

Kodomena: kada je $f(x,y) = c$ za neki proizvoljni c?

$x + 2y + 1 = c$

$x = c - 1 - 2y$

AKO ODABEREMO NEKI y, PA ZATIM $x = c - 1 - 2y$ TADA JE $f(x,y) = c$

DAKLE kodomena, $\exists$ slika $\mathrm{Im}(f) = \mathbb{R}$

RAVNINSKE KRIVULJE:

$f(x,y) = x + 2y + 1 = c$

$y = \frac{c - 1 - x}{2} = -\frac{1}{2}x + \frac{c-1}{2}$ PRAVCI

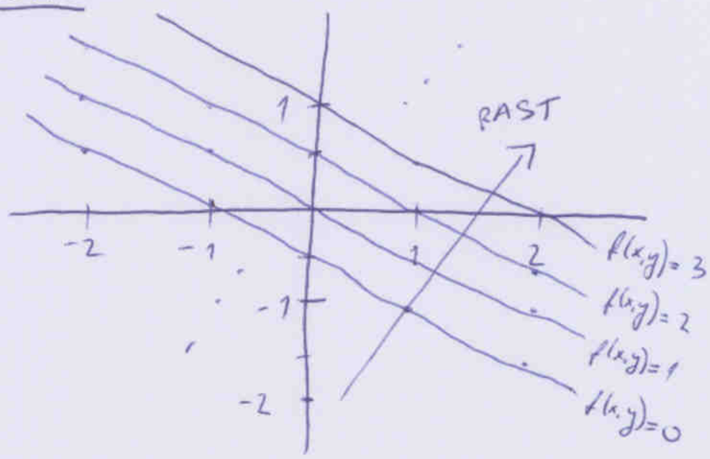
Za

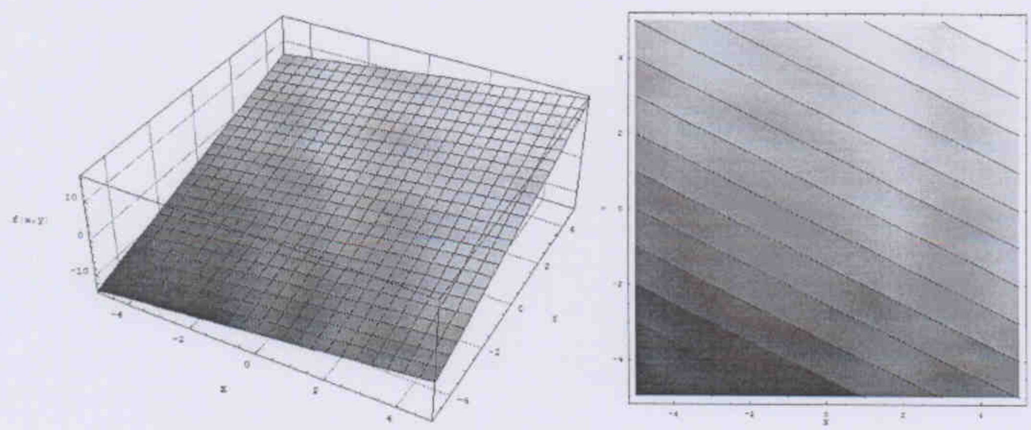
$c = 0 \quad y = -\frac{1}{2}x - \frac{1}{2}$

$c = 1 \quad y = -\frac{1}{2}x$

$c = 2 \quad y = -\frac{1}{2}x + \frac{1}{2}$

$c = 3 \quad y = -\frac{1}{2}x + 1$





Slika 2.0.5: $f(x, y) = x + 2y + 1$

Paraboloid: $z - z_0 = \dfrac{(x-x_0)^2}{A} + \dfrac{(y-y_0)^2}{B}$ $\quad \mathcal{D}(f) = \mathbb{R}\times\mathbb{R}$

Ispitati funkciju $f(x,y) = \underbrace{(x-1)^2}_{\geq 0} + \underbrace{y^2}_{\geq 0} - 9$.

ZBOG $(x-1)^2 \geq 0$, $y^2 \geq 0$ SLIJEDI $f(x,y) \geq -9$

ZA $c \geq -9$

$f(x,y) = c$ ZA $(x-1)^2 + y^2 - 9 = c$

$(x-1)^2 + y^2 = 9 + c$

$\Rightarrow \mathrm{Im}(f) = [-9, +\infty\rangle$

ZA $c = -9 \Rightarrow (x-1)^2 + y^2 = 0 \rightarrow$ TOČKA $T(1,0)$

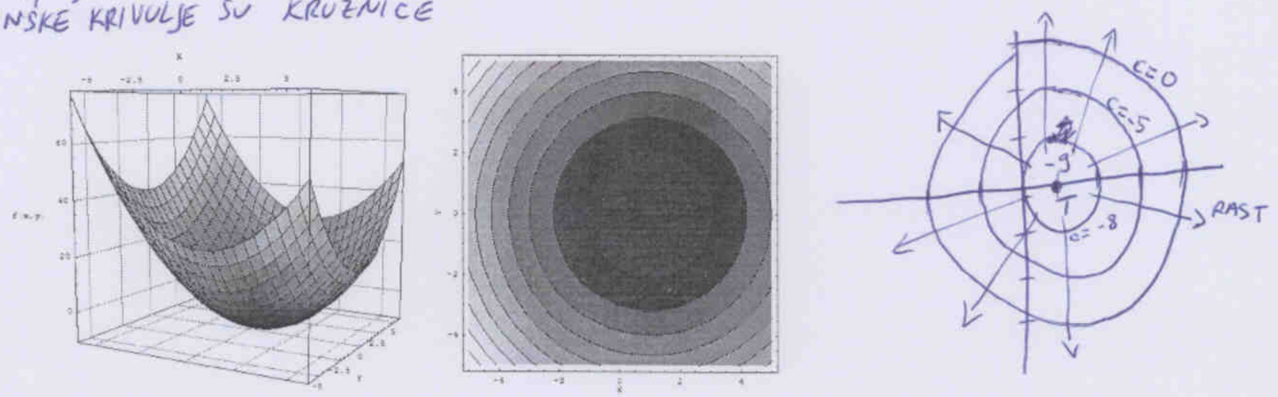
ZA $c = -8 \Rightarrow (x-1)^2 + y^2 = 1 \rightarrow$ KRUŽNICA $R=1$ OKO T

ZA $c = -5 \Rightarrow (x-1)^2 + y^2 = 4 \rightarrow$ -//- $R=2$ OKO T

ZA $c = 0 \Rightarrow (x-1)^2 + y^2 = 3^2 \rightarrow$ -//- $R=3$ OKO T

TO JE FORMULA ZA KRUŽNICU RADIJUSA $R = \sqrt{9+c}$ SA CENTROM U TOČKI $T(1,0)$

DAKLE, RAVNINSKE KRIVULJE SU KRUŽNICE



Slika 2.0.6: $f(x,y) = (x-1)^2 + y^2 - 9$

Ispitati funkciju $f(x,y) = (x-1)^2 - y^2$. $\quad \mathcal{D}(f) = \mathbb{R}\times\mathbb{R}$

MOŽE LI $f(x,y) = c$ ZA BILO KOJI c UZ POVOLJAN ODABIR x i y ?

$(x-1)^2 - y^2 = c$

$(x-1)^2 = c + y^2$

$x - 1 = \sqrt{c + y^2}$

$x = \sqrt{c + y^2} + 1$

AKO UZMEMO y TAKAV DA $c + y^2 \geq 0$, TO JEST $y^2 \geq -c$ I UZMEMO $x = \sqrt{c+y^2} + 1$ TADA JE $f(x,y) = c$

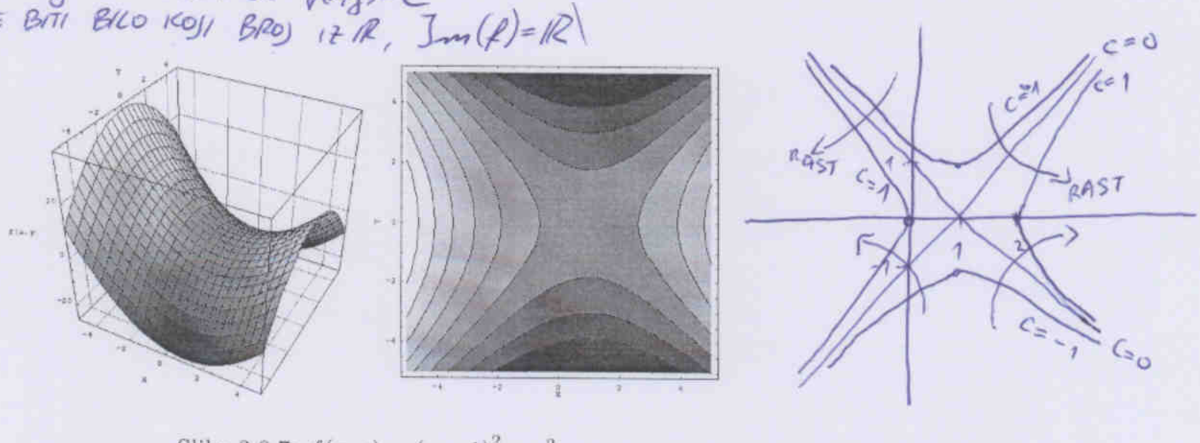
DAKLE, c MOŽE BITI BILO KOJI BROJ IZ $\mathbb{R}$, $\mathrm{Im}(f) = \mathbb{R}$

RAZINSKE KRIVULJE: $(x-1)^2 - y^2 = c$ SU HIPERBOLE

ZA $c = 0 \Rightarrow (x-1)^2 - y^2 = 0 \Rightarrow (x-1)^2 = y^2$
$\Rightarrow \left.\begin{array}{l} y = x - 1 \\ y = 1 - x \end{array}\right\}$ PRAVCI

ZA $c = 1$:
$(x-1)^2 - y^2 = 1$
$(x-1)^2 = y^2 + 1 \Rightarrow$ HIPERBOLA KROZ $(0,0)$, $(2,0)$

ZA $c = -1$
$(x-1)^2 = y^2 - 1 \Rightarrow$ HIPERBOLA KROZ $(1,1)$, $(1,-1)$



Slika 2.0.7: $f(x,y) = (x-1)^2 - y^2$

Stožac: $(z-z_0)^2=\left(\frac{x-x_0}{A}\right)^2+\left(\frac{y-y_0}{B}\right)^2$

Ispitati funkciju $f(x,y)=\sqrt{(x+2)^2+(y-1)^2}-3$.

IZRAZ POD KORIJENOM $(x+2)^2+(y-1)^2$ TREBA BITI VEĆI ILI JEDNAK NULI. ZBROJ KVADRATA JEST ≥ 0. $\Rightarrow D(f)=\mathbb{R}\times\mathbb{R}$

$f(x,y)=c$?

$\sqrt{(x+2)^2+(y-1)^2}-3=c$

$\underbrace{\sqrt{(x+2)^2+(y-1)^2}}_{\geq 0}=c+3 \;/^2$

KVADRIRANJE MOŽE ZBUNITI: $(x+2)^2+(y-1)^2=(c+3)^2$

$c+3\geq 0 \Rightarrow c\geq -3$

$\Rightarrow Im(f)=[-3,+\infty\rangle$

RAVNINSKE KRIVULJE:

KRUŽNICE

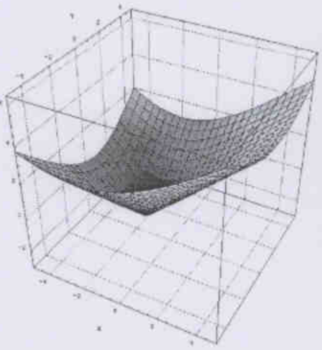
Pr. $\boxed{c=-3}$ $(x+2)^2+(y-1)^2=0 \Rightarrow T(-2,1)$

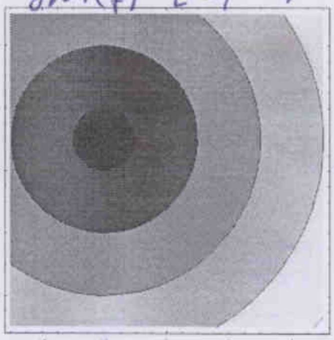
$\boxed{c=-2}$ $(x+2)^2+(y-1)^2=1 \Rightarrow K(T,1)$

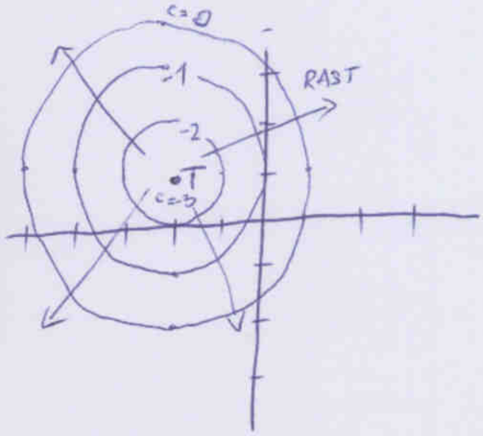
$\boxed{c=-1}$ $(x+2)^2+(y-1)^2=2 \Rightarrow K(T,2)$

$\boxed{c=0}$ $(x+2)^2+(y-1)^2=3 \Rightarrow K(T,3)$

ITD...







Slika 2.0.8: $f(x,y)=\sqrt{(x+2)^2+(y-1)^2}-3$

Elipsoid: $\left(\frac{x-x_0}{A}\right)^2+\left(\frac{y-y_0}{B}\right)^2+\left(\frac{z-z_0}{C}\right)^2=1$

Zadana je funkcija $f(x,y)=\sqrt{9-(x+2)^2 \cancel{+} -(y-1)^2}+1$. Odrediti domenu, kodomenu SLIKU i razinske krivulje.

IZRAZ POD KORIJENOM ≥ 0 :

$9-(x+2)^2-(y-1)^2\geq 0$

$\Rightarrow (x+2)^2+(y-1)^2\leq 9$

KRUG RADIJUSA $R=3$

$\Rightarrow D(f)=K(T(-2,1),3)$

$=\{(x,y):(x+2)^2+(y-1)^2\leq 9\}$

$f(x,y)=c$

$\sqrt{9-(x+2)^2-(y-1)^2}+1=c$

$\underbrace{\sqrt{9-(x+2)^2-(y-1)^2}}_{\geq 0}=c-1$

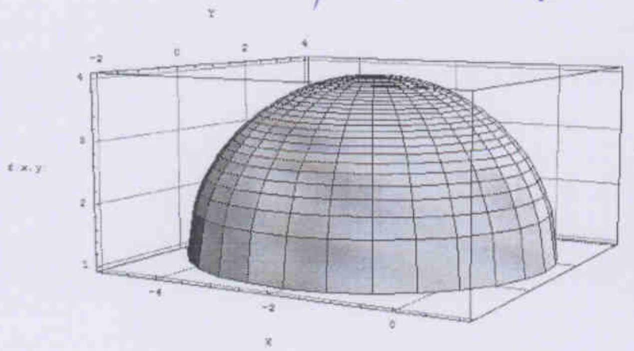
$\Rightarrow c-1\geq 0,\ c\geq 1$

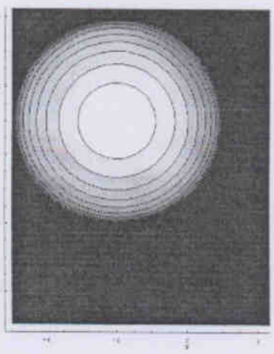
ZBOG $(x+2)^2+(y-1)^2\geq 0$ VRIJEDI

$9-(x+2)^2-(y-1)^2\leq 9 \Rightarrow \sqrt{9-(x+2)^2-(y-1)^2}\leq\sqrt{9}=3$

$\Rightarrow 3+1\geq c \Rightarrow c\leq 4$

$\boxed{Im(f)=[1,4]}$





Slika 2.0.9: $f(x,y)=\sqrt{9-(x+2)^2+(y-1)^2}+1$

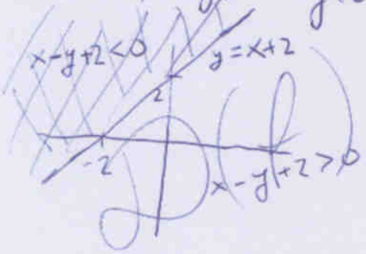
Primjer $f(x,y) = \ln(x-y+2)$

Odrediti domenu, kodomenu i razinske krivulje.

ARGUMENT OD ln TREBA BITI > 0

$\Rightarrow x-y+2>0$

$D(f) = \{(x,y) : x-y+2>0\}$



$Im(f) = ?$

$\ln(x-y+2) = c$

$\Rightarrow \underbrace{e^{\ln(x-y+2)}}_{=x-y+2} = e^c$

$y = \cancel{x+2} - e^c$

$\Rightarrow Im(f) = \mathbb{R}$

RAZINSKE KRIVULJE

$f(x,y) = c$

$\vdots$

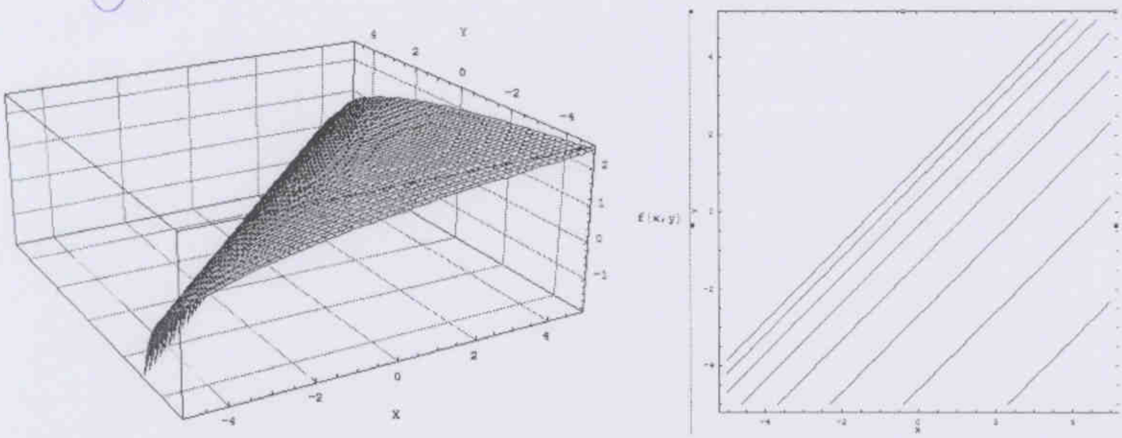
$y = x+2-e^c$

Pr. $c=0 \Rightarrow y = x+1$

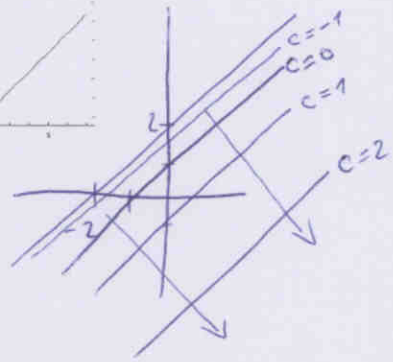
$c=1 \Rightarrow y = x+2-e \approx x-0.7$

$c=2 \Rightarrow y = x+2-e^2 \approx x-5.4$

$c=-1 \Rightarrow y = x+2-e^{-1} \approx x+1.63$



Slika 2.0.10: $f(x,y) = \ln(x-y+2)$



Zadatak za samostalan rad $f(x,y) = \ln(\frac{x}{y})$

Odrediti domenu, kodomenu i razinske krivulje.

Limes (granična vrijednost) i neprekidnost

Točke u višedimenzionalnom prostoru označavati ćemo sa masnim slovima. Npr. $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. Sa $K(\mathbf{x}, \delta)$ označavamo kuglu oko točke $\mathbf{x}$ radijusa δ.

Definicija limesa funkcije. Skalarna funkcija $f : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}$ ima limes L u točki $\mathbf{x} \in \mathbb{R}^m$

- ako za svaki $\varepsilon > 0$ postoji $\delta > 0$ tako da za svaki $\mathbf{y} \in \Omega$ takav da $\mathbf{y} \neq \mathbf{x}$ i $\mathbf{y} \in K(\mathbf{x}, \delta)$ vrijedi još i $|f(\mathbf{y}) - f(\mathbf{x})| < \varepsilon$,
- ako za svaki niz $\{\mathbf{y}_k : k \in \mathbb{N}\} \subseteq \Omega$ takav da $\mathbf{y}_k \to \mathbf{x}$ vrijedi $f(\mathbf{y}_k) \to L$.

Posljedica: funkcija nema limes na mjestima gdje se dodiruju ili križaju dvije različite razinske krivulje (ili razinske plohe)!

Definicija neprekidnosti. Funkcija $f : \Omega \to \mathbb{R}$ je neprekidna ako u svakoj točki $\mathbf{x} \in \Omega$ postoji limes funkcije i jednak je $f(\mathbf{x})$.

Primjer $f(x, y) = \dfrac{x}{y}$

Odrediti domenu, kodomenu, razinske krivulje i limes u ishodištu (ako postoji).

$\mathcal{D}(f) = \{(x,y) : y \neq 0\}$

$\operatorname{Im}(f) = ?$

$f(x,y) = c$?

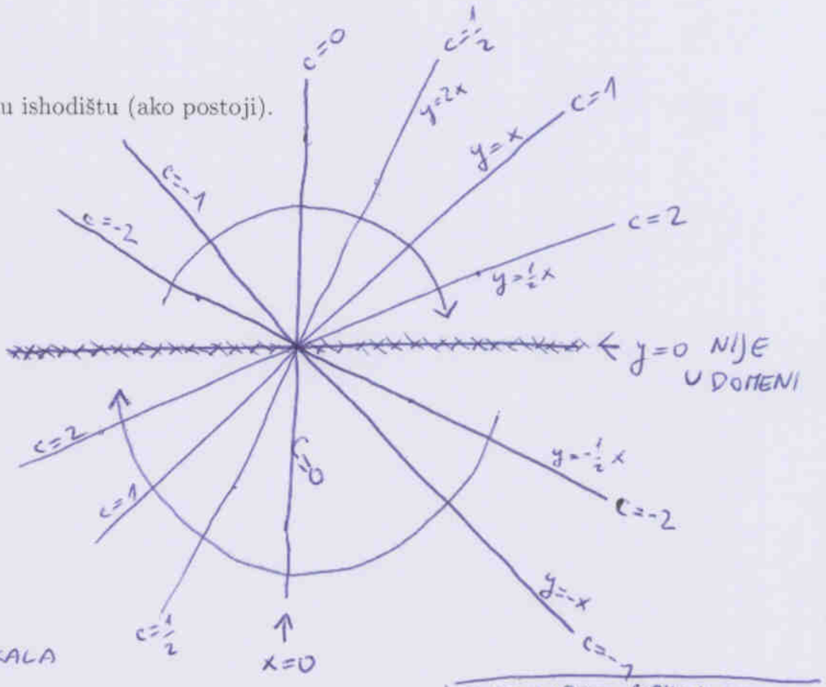
$\frac{x}{y} = c$

$x = c \cdot y \Rightarrow \operatorname{Im}(f) = \mathbb{R}$

RAZINSKE KRIVULJE

$f(x,y) = c$

$\frac{x}{y} = c$



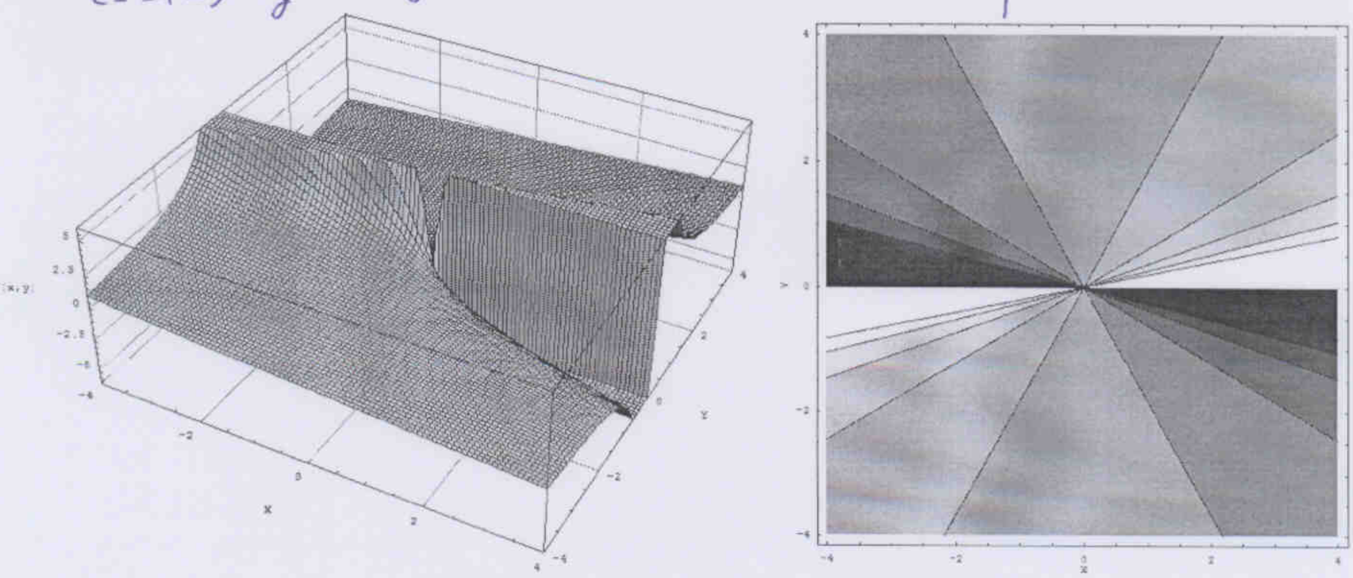
Pr. $c=0 \Rightarrow \frac{x}{y}=0 \Rightarrow x=0$ VERTIKALA

$c=1 \Rightarrow \frac{x}{y}=1 \Rightarrow y=x$ $\qquad c=\frac{1}{2} \Rightarrow \frac{x}{y}=\frac{1}{2} \Rightarrow y=2x$

$c=2 \Rightarrow \frac{x}{y}=2 \Rightarrow y=\frac{1}{2}x$ $\qquad c=-2 \Rightarrow \frac{x}{y}=-2 \Rightarrow y=-\frac{1}{2}x$

$c=-1 \Rightarrow \frac{x}{y}=-1 \Rightarrow y=-x$

RAZINSKE KRIVULJE SIJEKU SE U ISHODIŠTU DAKLE, NEMA LIMESA U ISHODIŠTU.



Slika 2.0.11: $f(x, y) = \frac{x}{y}$